

# LEVI SUBGROUP ACTIONS ON SCHUBERT VARIETIES, INDUCED DECOMPOSITIONS OF THEIR COORDINATE RINGS, SPHERICITY AND SINGULARITY CONSEQUENCES

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**ABSTRACT.** Let  $L_w$  be the Levi part of the stabilizer  $Q_w$  in  $GL_N(\mathbb{C})$  (for left multiplication) of a Schubert variety  $X(w)$  in the Grassmannian  $G_{d,N}$ . For the natural action of  $L_w$  on  $\mathbb{C}[X(w)]$ , the homogeneous coordinate ring of  $X(w)$  (for the Plücker embedding), we give a combinatorial description of the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules; in fact, our description holds more generally for the action of the Levi part  $L$  of any parabolic group  $Q$  that is a subgroup of  $Q_w$ . Using this combinatorial description, we give a classification of all Schubert varieties  $X(w)$  in the Grassmannian  $G_{d,N}$  for which  $\mathbb{C}[X(w)]$  has a decomposition into irreducible  $L_w$ -modules that is multiplicity free. This classification is then used to show that certain classes of Schubert varieties are spherical  $L_w$ -varieties. These classes include all smooth Schubert varieties, all determinantal Schubert varieties, as well as all Schubert varieties in  $G_{2,N}$  and  $G_{3,N}$ . Also, as an important consequence, we get interesting results related to the singular locus of  $X(w)$  and multiplicities at  $T$ -fixed points in  $X(w)$ .

## 1. INTRODUCTION

Let  $GL_N(\mathbb{C})$  be the group of invertible  $N \times N$  matrices over  $\mathbb{C}$ . Let  $B$  be the Borel subgroup of  $G$  consisting of upper triangular matrices, and  $T$  the maximal torus consisting of diagonal matrices. For  $1 \leq d \leq N-1$ , let  $G_{d,N}$  denote the Grassmannian variety consisting of  $d$ -dimensional subspaces of  $\mathbb{C}^N$ . The  $T$ -fixed points in  $G_{d,N}$  are denoted by  $[e_w]$  for  $w \in I_{d,N} := \{\underline{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq N\}$ . The Schubert varieties in  $G_{d,N}$  are the Zariski closures of  $B$ -orbits through the  $T$ -fixed points, that is for  $w \in I_{d,N}$  the Schubert variety  $X(w) := \overline{B[e_w]}$ . We denote the homogeneous coordinate ring of  $X(w)$  for the Plücker embedding by  $\mathbb{C}[X(w)]$ .

Let  $G$  be a reductive group with  $B_G$  a Borel subgroup. Suppose that  $X$  is an irreducible  $G$ -variety, then  $X$  is a spherical  $G$ -variety if it is normal and has a dense open  $B_G$ -orbit.

Our initial goal was to understand when a Schubert variety  $X(w)$  is spherical for the left multiplication action of reductive subgroups of  $GL_N(\mathbb{C})$  that stabilize  $X(w)$ . Using Proposition 5.0.1 we relate this sphericity question to the module structure of  $\mathbb{C}[X(w)]$  under the induced action of these reductive subgroups.

Fix a  $w \in I_{d,N}$ . There is a canonical choice of reductive subgroups of  $GL_N(\mathbb{C})$  that stabilize  $X(w)$ . Let  $Q_w$  be the stabilizer in  $GL_N(\mathbb{C})$  of  $X(w)$ ; it is clearly a parabolic subgroup of  $GL_N(\mathbb{C})$ . Let  $L_w$  be the Levi part of  $Q_w$ , it is a reductive group. We have a natural action of  $L_w$  on  $\mathbb{C}[X(w)]$ . The main result of this paper is an explicit description of the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -submodules. In fact, our description holds for a much larger class of reductive subgroups, that is, for the Levi part  $L$  of any parabolic subgroup  $Q \subseteq Q_w$  (cf. Theorem 3.5.3 and Corollary 3.5.9). Though it would be enough to give such a decomposition for  $L_w$  and deduce the result for any  $L \subseteq L_w$  by using branching rules, our procedure is independent of the choice of  $L$ ; further, using our description for any  $L$  we are able to deduce interesting branching rule formulas (cf. Remark 3.5.10).

Our proof uses the standard monomial basis for  $\mathbb{C}[X(w)]$ . As a graded ring, we have that  $\mathbb{C}[X(w)]_r, r \in \mathbb{N}$  has a vector space basis given by the set  $\text{Std}_r$  of all standard monomials of degree  $r$ . We give the decomposition for  $\mathbb{C}[X(w)]_r$  as an  $L$ -module, which we describe briefly below, in terms of certain Weyl modules associated to  $L$ .

Given  $X(w)$ , and a Levi subgroup  $L$  as above, our first step involves capturing certain Schubert subvarieties  $X(\theta)$ , characterized by the property that  $L$  is the Levi part of  $Q_\theta$ , the stabilizer in  $\text{GL}_N(\mathbb{C})$  of  $X(\theta)$ . A combinatorial description of all such  $\theta \in H_w := \{\tau \in W^P \mid \tau \leq w\}$  is given in Proposition 3.1.5. We refer to these  $\theta$  as *the heads of type  $L$*  and denote the subset of heads of type  $L$  in  $H_w$  by  $\text{Head}_L$ . The critical part of this step is showing how  $L$  gives rise to a nice partition of the Hasse diagram of  $H_w$  into disjoint subdiagrams, each containing a unique head of type  $L$ . Then for a  $\tau \in H_w$  we define  $\theta_\tau \in \text{Head}_L$  to be the unique head in the disjoint subdiagram containing  $\tau$ . Finally for  $\theta \in \text{Head}_L$  we define  $\text{WStd}_\theta := \{\tau \in H_w \mid \theta_\tau = \theta\}$  and  $\text{Std}_\theta := \{p_\tau \in \text{Std}_1 \mid \tau \in \text{WStd}_\theta\}$ . Thus  $\text{WStd}_\theta$  is the collection of all elements connected to  $\theta$  in the disjoint Hasse diagram. This gives us the decompositions

$$\text{Std}_1 = \bigsqcup_{\theta \in \text{Head}_L} \text{Std}_\theta$$

and

$$\mathbb{C}[X(w)]_1 = \langle \text{Std}_1 \rangle = \bigoplus_{\theta \in \text{Head}_L} \langle \text{Std}_\theta \rangle.$$

This decomposition of  $\mathbb{C}[X(w)]_1$  would hold for any partition of the Hasse diagram of  $H_w$  into disjoint subdiagrams each containing a unique head of type  $L$ . However, for the partition we consider we have that the  $\langle \text{Std}_\theta \rangle$  are in fact irreducible  $L$ -submodules; our partition is the unique partition for which this is the case. Thus the above decomposition is an  $L$ -module decomposition of the degree one part of  $\mathbb{C}[X(w)]$ .

The second step is to extend this idea to higher degrees.

For  $\tau_1, \dots, \tau_r \in H_w$  we define the degree  $r$  head of  $(\tau_1, \dots, \tau_r)$  to be the sequence  $(\theta_{\tau_1}, \dots, \theta_{\tau_r})$ . Then  $\text{Head}_{L,r}$  is defined to be the set of degree  $r$  heads  $(\theta_1, \dots, \theta_r)$  such that  $\theta_1 \geq \dots \geq \theta_r$ . As before for  $\underline{\theta} \in \text{Head}_{L,r}$  we define  $\text{Std}_{\underline{\theta}} := \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid \underline{\theta} = (\theta_{\tau_1}, \dots, \theta_{\tau_r})\}$ . Remarkably these  $\text{Std}_{\underline{\theta}}$  for  $\underline{\theta} \in \text{Head}_{L,r}$  once again partition the set  $\text{Std}_r$ . This is by no means readily apparent and is due to the fact that given  $\tau_1, \dots, \tau_r \in H_w$  such that  $\tau_1 \geq \dots \geq \tau_r$  we have  $\theta_{\tau_1} \geq \dots \geq \theta_{\tau_r}$  (cf. Proposition 3.2.2). Thus we have the decompositions

$$\text{Std}_r = \bigsqcup_{\underline{\theta} \in \text{Head}_{L,r}} \text{Std}_{\underline{\theta}}$$

and

$$\mathbb{C}[X(w)]_r = \langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} \langle \text{Std}_{\underline{\theta}} \rangle.$$

Unfortunately, the latter decomposition is no longer a  $L$ -module decomposition. This is due to the way that the  $L$ -action interacts with the standard monomial straightening rule which results in certain  $\langle \text{Std}_{\underline{\theta}} \rangle$  not being  $L$ -stable. Thus in higher degrees our decomposition must be modified.

To achieve this we introduce the partial order  $\geq_{str}$  on the set of degree  $r$  heads  $\text{Head}_{L,r}$ . For  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$  we define

$$\text{Std}_{\underline{\theta}}^{\geq_{str}} = \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid \underline{\theta} \geq_{str} (\theta_{\tau_1}, \dots, \theta_{\tau_r})\}$$

and

$$\text{Std}_{\underline{\theta}}^{>_{str}} = \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid \underline{\theta} >_{str} (\theta_{\tau_1}, \dots, \theta_{\tau_r})\}.$$

The next step is to show that  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  and  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  are  $L$ -stable. To show this we equivalently check that they are  $\text{Lie}(L)$  stable. Once we have done this we define  $U_{\underline{\theta}}$  to be a  $L$ -module complement of  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  inside of  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$ .

A final result is required before the description of the degree  $r$  decomposition. The  $U_{\underline{\theta}}$  are  $L$ -submodules of  $\langle \text{Std}_r \rangle$ . As  $L$  is equal to a product of general linear groups we should be able to describe any  $L$ -module, in particular the  $U_{\underline{\theta}}$ , in terms of tensor products of Weyl modules. As vector spaces  $U_{\underline{\theta}} \cong \langle \text{Std}_{\underline{\theta}} \rangle$ . In view of this we first describe a vector space map from  $\langle \text{Std}_{\underline{\theta}} \rangle$  to a certain tensor product of (skew) Weyl modules denoted  $W_{\underline{\theta}}$ , using the combinatorics of the standard monomials. This map as well as some character arguments are then used to conclude that the  $L$ -module  $U_{\underline{\theta}}$  has the form  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$ , where  $D_r$  is a tensor product of certain determinant representations.

We are now ready to state our main result(cf. Theorem 3.5.3):

**Theorem 1.1.** *Let  $\underline{\theta} \in \text{Head}_{L,r}$ . There exists a  $L$ -module  $U_{\underline{\theta}}$  such that we have the following  $L$ -module isomorphisms:*

- (a)  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle = U_{\underline{\theta}} \oplus \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$ .
- (b)  $\langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} U_{\underline{\theta}}$ .
- (c)  $U_{\underline{\theta}} \cong \mathbb{W}_{\underline{\theta}}^* \otimes D_r$

The  $U_{\underline{\theta}}$  may not be irreducible  $L$ -modules, but their decomposition into irreducibles can now be computed simply by calculating the decomposition of certain tensor products of Weyl modules. This is done in Corollary 3.5.9 where we give the explicit decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L$ -modules.

The above decomposition may then be applied to the classification of those Schubert varieties whose coordinate rings have a multiplicity free decomposition into irreducible  $L$ -modules. To make the statement of our multiplicity results more tractable we restrict our discussion to  $L = L_w$ , though our techniques are applicable for any  $L$ . With the classification accomplished we then show that the class of Schubert varieties whose coordinate rings have a multiplicity free decomposition into irreducible  $L_w$ -modules includes all smooth Schubert varieties, all determinantal Schubert varieties(cf. Definition 4.0.21), as well as all Schubert varieties in  $G_{2,N}$  and  $G_{3,N}$ .

We note that if  $\mathbb{C}[X(w)]$  has a multiplicity free decomposition into irreducible  $L_w$ -modules then  $\widehat{X(w)}$ , the cone over  $X(w)$ , is a spherical  $L_w$ -variety. We then use this to show that  $X(w)$  is a spherical  $L_w$ -variety. We conclude that all smooth Schubert varieties, all determinantal schubert varieties(and determinantal varieties), as well as all Schubert varieties in  $G_{2,N}$  and  $G_{3,N}$  are spherical  $L_w$ -varieties. We also get that the coordinate ring of any determinantal variety has a multiplicity free decomposition into irreducible  $L_w$ -modules.

As a further important consequence we get some interesting results relating the singularities of  $X(w)$  and the degree 1 heads  $\theta \in \text{Head}_{L_w}$ . We first give a description of the singular locus of  $X(w)$  in terms of maximal degree 1 heads (cf. Corollary 6.1.5(a)). Using this, we show that the set of  $T$ -fixed points in the smooth locus of  $X(w)$  is precisely  $\{[e_{\tau}], \tau \in \text{WStd}_w\}$  (cf. Corollary 6.2.6(a)); here,  $\text{WStd}_w$  is the set of all elements of  $H_w$  connected to  $w$  in the disjoint Hasse diagram. Further, we prove that all elements in  $\text{WStd}_{\theta}$ (the set of all elements of  $H_w$  connected to  $\theta$  in the disjoint Hasse diagram) have associated  $T$ -fixed points occurring with the same multiplicity in  $X(w)$  (cf. Proposition 6.2.7). Finally, we show that the set of  $T$ -fixed points in the smooth locus of  $X(\theta)$  contains  $\{[e_{\tau}], \tau \in \text{WStd}_{\theta}\}$ , with equality under certain conditions on  $Q_w$  and  $Q_{\theta}$  (cf. Corollary

6.2.9). Note that a reader interested in these singularity results need only read Section 3.1 and Section 6.

Thus this paper is really at the crossroads of representation theory, combinatorics, and geometry. We hope to extend the results of this paper, using similar techniques, to any Schubert variety in  $GL_N/Q$ , where  $Q$  is any parabolic subgroup, as well as to Schubert varieties in the Lagrangian and Orthogonal Grassmannians. The combinatorial results that one may obtain for the spherical Schubert varieties (by virtue of them being spherical varieties) should also be interesting. We plan to investigate such combinatorial results in a subsequent paper.

The sections are organized as follows: Section 2 is on Preliminaries pertaining to Schubert varieties in  $G_{d,N}$ , standard monomial basis, and representation theory of the general linear group. In Section 3, we introduce the heads of type  $L$ , the degree  $r$  heads, and after proving some preparatory results, we determine the decomposition (as an  $L$ -module) of  $\mathbb{C}[X(w)]$ . In Section 4, we give the classification of the  $X(w)$  for which the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -submodules is multiplicity-free. In Section 5, we apply the classification results to deduce the sphericity of smooth and determinantal Schubert varieties, as well as the sphericity of all Schubert varieties in  $G_{2,n}, G_{3,n}$ . In Section 6, we prove numerous results regarding the singular locus of  $X(w)$ , and the multiplicity in  $X(w)$  of the  $T$ -fixed points.

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## 2. PRELIMINARIES

**2.1. Standard Monomial Theory.** In this section we fix the notation that will be used throughout the paper. For a more in depth introduction to these topics both [LB15] and [LR08] may be consulted.

Fix a positive integer  $N$ , and let  $\{e_1, \dots, e_N\}$  be the standard basis of  $\mathbb{C}^N$ . We will do all computations over the field  $\mathbb{C}$ . We denote by  $GL_N$  the invertible  $N \times N$  matrices over  $\mathbb{C}$ . Let  $T$  be the *standard maximal torus* comprised of diagonal matrices, and  $B$  the *standard Borel subgroup* comprised of upper triangular matrices.

Let  $X(T) := \text{Hom}_{\text{alg.grp}}(T, \mathbb{C}^*)$  be the character group of  $T$ ; it is a free abelian group of rank  $N$  with a basis  $\{\epsilon_i, 1 \leq i \leq N\}$ ,  $\epsilon_i$  being the character which sends a diagonal matrix  $\text{diag}(t_1, \dots, t_N)$  to its  $i$ -th entry  $t_i$ . The elements of  $X(T)$  will be referred to (formally) as *weights*. We will often simplify our notation by referring to an element of  $X(T)$  by the sequence  $(a_1, \dots, a_N)$ ,  $a_i \in \mathbb{Z}$ , which corresponds to the weight  $\sum a_i \epsilon_i \in X(T)$ .

A weight  $(a_1, \dots, a_N)$  such that  $a_1 \geq \dots \geq a_N \geq 0$  is called a *dominant weight* (cf. [FH91]). Recall that the set of all dominant weights gives an indexing of the set of all irreducible polynomial representations of  $GL_N$ .

Let  $V$  be a finite-dimensional  $T$ -module. Then we have the decomposition

$$V = \bigoplus_{\chi \in X(T)} V_\chi$$

where  $V_\chi$  is  $T$ -weight space consisting of all vectors  $v \in V$  such that  $t \cdot v = \chi(t)v$ , for all  $t \in T$ . If  $v \in V_\chi$  we say that  $v$  has weight  $\chi$ , and write  $wt(v) = \chi$ . Let  $m_\chi = \dim V_\chi$ . We define the

character of  $V$ , denoted  $\text{char}(V)$ , as the element in  $\mathbb{Z}[X(T)]$ , the group algebra of  $X(T)$ , given by

$$\text{char}(V) := \sum m_\chi e^\chi.$$

**Remark 2.1.1.** If  $G$  is a product, say  $G = GL_M \times GL_N$ , by a dominant weight of  $G$ , we shall mean a sequence  $(a_1, \dots, a_M, b_1, \dots, b_N)$ ,  $a_i, b_j \in \mathbb{Z}$ , with  $a_1 \geq \dots \geq a_M \geq 0$  and  $b_1 \geq \dots \geq b_N \geq 0$ . Given a  $GL_M$ -module  $V$  and a  $GL_N$ -module  $W$ , consider the  $GL_M \times GL_N$ -module  $V \otimes W$  given by the natural diagonal action; we say that  $V \otimes W$  is a polynomial representation of  $GL_M \times GL_N$ , if  $V, W$  are polynomial representations of  $GL_M, GL_N$  respectively. By  $\text{char}(V \otimes W)$ , we shall mean the element  $(\text{char } V, \text{char } W)$  of  $\mathbb{Z}[X(T_M)] \times \mathbb{Z}[X(T_N)]$ , where  $T_M, T_N$  denote the maximal tori, consisting of diagonal matrices in  $GL_M, GL_N$  respectively. More generally for  $G = GL_{N_1} \times \dots \times GL_{N_r}$  these notions extend in the obvious way.

Let  $\Phi$  be the *root system* of  $GL_N$  (cf. [LR08, Chapter 3]). It is the set  $\{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq N\}$ , where  $\epsilon_i - \epsilon_j$  is the element of  $X(T)$  which sends  $\text{diag}(t_1, \dots, t_n)$  in  $T$  to  $t_i t_j^{-1}$  in  $\mathbb{C}$ . Our choice of the torus  $T$  and the Borel subgroup  $B$  induces a set of *positive roots*  $\Phi^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq N\}$  and *simple roots*  $S = \{\alpha_i := \epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq N-1\}$ .

The Weyl group  $W$  of  $GL_N$  is generated by the simple reflections  $s_{\alpha_i}$  for  $\alpha_i \in S$  and is isomorphic to the symmetric group  $S_N$  of permutations on  $N$  symbols under the map sending  $s_{\alpha_i}$  to the transposition  $(i, i+1)$ . For every  $1 \leq d \leq N-1$  there is a *maximal parabolic subgroup*  $P_{\hat{d}}$  of  $GL_N$  that corresponds to the subgroup of all matrices with a block of size  $N-d \times d$  in the lower left corner with all entries equal to zero.

$$P_{\hat{d}} = \left\{ \begin{bmatrix} * & * \\ 0_{N-d \times d} & * \end{bmatrix} \in GL_N \right\}$$

**Remark 2.1.2.** There is a bijection between the subsets  $A \subset \{1, \dots, N-1\}$  and the parabolic subgroups of  $GL_N$ , given by

$$P_A = \bigcap_{d \in \{1, \dots, N-1\} \setminus A} P_{\hat{d}}.$$

The one-line notation for elements of  $W$  is a sequence  $(a_1, \dots, a_N)$  and corresponds to the permutation that sends  $i \mapsto a_i$ . For the parabolic subgroup  $P_A$ ,  $A \subset \{1, \dots, N-1\}$ ,  $W_{P_A}$  is the subgroup of  $W$  generated by  $\{s_{\alpha_i} \mid i \in A\}$ . And  $W^{P_A}$  is the subset of  $W$  made up of minimal representatives, under the Bruhat order, in  $W$  of elements of  $W/W_{P_A}$ . For all  $1 \leq d \leq N-1$  we have that  $W^{P_{\hat{d}}}$  can be represented as the set  $\{(a_1, \dots, a_d) \mid 1 \leq a_1 < \dots < a_d \leq N\}$ .

The *Grassmannian*  $G_{d,N}$  is the set of all  $d$ -dimensional subspaces of  $\mathbb{C}^N$ . For  $U \in G_{d,N}$  fix a basis  $\{u_1, \dots, u_d\}$  of  $U$  and define the map

$$\begin{aligned} G_{d,N} &\longrightarrow \mathbb{P}(\wedge^d \mathbb{C}^N) \\ U &\mapsto [u_1 \wedge \dots \wedge u_d] \end{aligned}$$

This is the well-known Plücker embedding. Set  $I_{d,N} := \{\underline{i} = (i_1, \dots, i_d) \mid 1 \leq i_1 < \dots < i_d \leq N\}$ . Then  $\{e_{\underline{i}} := e_{i_1} \wedge \dots \wedge e_{i_d}\}_{\underline{i} \in I_{d,N}}$  is the standard basis for  $\wedge^d \mathbb{C}^N$ . Let  $\{p_{\underline{j}}\}_{\underline{j} \in I_{d,N}}$  be the basis for  $(\wedge^d \mathbb{C}^N)^*$  dual to  $\{e_{\underline{i}}\}_{\underline{i} \in I_{d,N}}$ . This dual basis gives a set of projective coordinates for  $\mathbb{P}(\wedge^d \mathbb{C}^N)$ , called the *Plücker coordinates*. These coordinates have a particularly nice description for points in  $G_{d,N}$  in terms of determinants: for  $U \in G_{d,N}$  fix a basis  $\{u_1, \dots, u_d\}$  of  $U$  as above and let  $A$  be the  $N \times d$  matrix with columns  $u_1, \dots, u_d$ . Then the Plücker coordinate  $p_{\underline{j}}(U)$  is the  $d$ -minor with row indices  $j_1, \dots, j_d$  of  $A$ . The Plücker embedding equips  $G_{d,N}$  with a projective variety structure, realized as the zero set of the well known quadratic Plücker relations.

The Grassmannian is a homogeneous space for the action of  $GL_N$  induced on  $\mathbb{P}(\wedge^d \mathbb{C}^N)$  by the  $GL_N$  action on  $\mathbb{C}^N$  and hence  $\wedge^d \mathbb{C}^N$ . Let  $e_{id} := e_1 \wedge \dots \wedge e_d$ , then  $[e_{id}] \in \mathbb{P}(\wedge^d \mathbb{C}^N)$  and the  $GL_N$

orbit through  $[e_{id}]$  is  $G_{d,N}$ , and the isotropy subgroup at  $[e_{id}]$  is precisely  $P_d^\wedge$ . Thus we obtain the identification  $G_{d,N} = GL_N/P_d^\wedge$ .

It is no coincidence that  $I_{d,N} = W^{P_d^\wedge}$ . The  $T$ -fixed points in  $GL_N/P_d^\wedge$  are  $\{wP_d^\wedge\}$  for  $w \in W^{P_d^\wedge}$ . Under the identification of  $GL_N/P_d^\wedge$  with  $G_{d,N}$  we see that  $\{wP_d^\wedge\}$  gets identified with the point  $[e_{i_1} \wedge \cdots \wedge e_{i_d}]$  where  $w$  is the element of  $W$  with first  $d$  entries equal to  $\{i_1, \dots, i_d\}$ , which corresponds to  $(i_1, \dots, i_d) \in W^{P_d^\wedge}$ . Thus the  $T$ -fixed points of  $G_{d,N}$  are  $[e_{i_1} \wedge \cdots \wedge e_{i_d}]$  for  $\underline{i} \in I_{d,N} = W^{P_d^\wedge}$ . It will prove useful to emphasize the homogeneous space identification of  $G_{d,N}$  and so subsequently we shall use  $W^{P_d^\wedge}$  to index the  $T$ -fixed points as well as the Plücker coordinates.

**Remark 2.1.3.** Let  $\tau \in W^{P_d^\wedge}$ , say  $\tau = (i_1, \dots, i_d)$ . Denote the integers  $\{1, \dots, N\} \setminus \{i_1, \dots, i_d\}$  by  $j_1, \dots, j_{N-d}$  (arranged in ascending order). We identify the Plücker co-ordinate  $p_\tau$  with the element  $e_{j_1} \wedge \cdots \wedge e_{j_{N-d}}$  in  $\bigwedge^{N-d} \mathbb{C}^N$ . Thus, the weight of  $p_\tau$  is  $\epsilon_{j_1} + \cdots + \epsilon_{j_{N-d}}$ . The weight is given by the sequence  $\chi_\tau := (\chi_1, \dots, \chi_N)$  where

$$\chi_i := \begin{cases} 0 & i \in \tau \\ 1 & i \notin \tau \end{cases}$$

for all  $1 \leq i \leq N$ .

For  $w \in W^{P_d^\wedge}$  the *Schubert variety* in  $G_{d,N}$  associated to  $w$  is  $X_{P_d^\wedge}(w) := \overline{B \cdot [e_w]} = \overline{BwP_d^\wedge/P_d^\wedge}$ , the  $B$  orbit closure through the  $T$ -fixed point  $[e_w]$ , equipped with the canonical reduced scheme structure.

There is a natural partial order on  $W^{P_d^\wedge}$ , referred to as the *Bruhat order*, induced by the partial order on the set of Schubert varieties given by inclusion. For  $\tau := (i_1, \dots, i_d), w := (j_1, \dots, j_d) \in W^{P_d^\wedge}$  we have  $\tau \leq w$  if and only if  $i_1 \leq j_1, \dots, i_d \leq j_d$ . Then  $[e_\tau] \in X(w)$  if and only if  $X(\tau) \subseteq X(w)$  and this is if and only if  $\tau \leq w$ .

We have the *Bruhat decomposition*

$$X(w) = \bigcup_{\tau \leq w} B[e_\tau].$$

Note that for the choice of  $w = (N-d+1, \dots, N) \in W^{P_d^\wedge}$  we have that the Schubert variety  $X(w)$  is in fact  $G_{d,N}$  itself.

Now consider the projective embedding

$$X(w) \hookrightarrow G_{d,N} \hookrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^N).$$

Let  $\mathbb{C}[X(w)]$  be the homogeneous coordinate ring of  $X(w)$  for this projective embedding. As a  $\mathbb{C}$ -algebra it is generated by  $p_\tau, \tau \leq w$ . This follows from the fact that  $p_\tau([e_w]) = \delta_{\tau,w}$ , which implies that  $p_\tau|_{X(w)} \not\equiv 0$  if and only if  $[e_\tau] \in X(w)$ , which occurs if and only if  $\tau \leq w$ . Thus for  $\tau_1, \dots, \tau_r, w \in W^{P_d^\wedge}$  with  $\tau_i \leq w$  for all  $1 \leq i \leq r$ , we have  $p_{\tau_1} \cdots p_{\tau_r} \in \mathbb{C}[X(w)]_r$ .

**Definition 2.1.4.** We define the monomial  $p_{\tau_1} \cdots p_{\tau_r}$  to be *standard* if  $\tau_1 \geq \cdots \geq \tau_r$ . It is *standard on  $X(w)$*  if in addition  $w \geq \tau_1$ .

**Theorem 2.1.5.** (cf. [LR08, Theorem 4.3.3.2]) *Monomials of degree  $r$  standard on  $X(w)$  give a  $\mathbb{C}$ -basis for  $\mathbb{C}[X(w)]_r$ .*

**2.2. The Straightening Algorithm.** The generation portion of Theorem 2.1.5 usually relies on exhibiting an inductive process that takes a nonstandard monomial and writes it as a sum of standard monomials. This is called straightening the nonstandard monomial, and the entire process is referred to as the straightening process.

The straightening process on the Grassmannian is comprised of an inductive step usually referred to as a shuffle. Let  $\tau := (i_1, \dots, i_d), \phi := (j_1, \dots, j_d) \in W^{P_d^\wedge}$  with  $\tau \not\geq \phi$ , that is  $p_\tau p_\phi$  is not standard.



This implies there exists a  $t$ ,  $t \leq d$  such that  $i_m \geq j_m$ , for all  $1 \leq m \leq t-1$ , and  $i_t < j_t$ . Let  $[\tau, \phi]$  denote the set of permutations  $\sigma_1$ , *other than the identity permutation*, of the multiset  $\{i_1, \dots, i_t, j_t, \dots, j_d\}$  such that  $\sigma_1(i_1) < \dots < \sigma_1(i_t)$  and  $\sigma_1(j_t) < \dots < \sigma_1(j_d)$ .

Define  $\alpha^{\sigma_1} := (\sigma_1(i_1), \dots, \sigma_1(i_t), i_{t+1}, \dots, i_d) \uparrow$  and  $\beta^{\sigma_1} := (j_1, \dots, j_{t-1}, \sigma_1(j_t), \dots, \sigma_1(j_d)) \uparrow$  (here, for a  $d$ -tuple  $(l_1, \dots, l_d)$ ,  $(l_1, \dots, l_d) \uparrow$  denotes the  $d$ -tuple obtained from  $(l_1, \dots, l_d)$  by arranging the entries in ascending order). Then

$$p_\tau p_\phi = \sum_{\sigma_1 \in [\tau, \phi]} \pm p_{\alpha^{\sigma_1}} p_{\beta^{\sigma_1}}.$$

Note that it is possible to keep track of the signs in the above summation but we omit this step since it is not needed for our consideration. It is not difficult to see that either  $\alpha^{\sigma_1} = 0$ , due to a repeated entry, or  $\alpha^{\sigma_1} > \tau$ . For the same reasons either  $\beta^{\sigma_1} = 0$ , due to a repeated entry, or  $\beta^{\sigma_1} < \phi$ . We will refer to this as the *ordering property of the shuffle*.

A single shuffle is not always sufficient to straighten the monomial  $p_\tau p_\phi$ . It may be the case that for a  $\sigma_1 \in [\tau, \phi]$  the monomial  $p_{\alpha^{\sigma_1}} p_{\beta^{\sigma_1}}$  is not standard. And so we must apply a shuffle to  $p_{\alpha^{\sigma_1}} p_{\beta^{\sigma_1}}$ .

Suppose  $\alpha^{\sigma_1} = (k_1, \dots, k_d)$  and  $\beta^{\sigma_1} = (l_1, \dots, l_d)$ . Since  $\alpha^{\sigma_1} \not\leq \beta^{\sigma_1}$ , there is a  $t'$ ,  $t' \leq d$  such that  $k_m \geq l_m$ , for all  $1 \leq m \leq t'-1$ , and  $k_{t'} < l_{t'}$ . Then for a  $\sigma_2 \in [\alpha^{\sigma_1}, \beta^{\sigma_1}]$  we define  $(\alpha^{\sigma_1})^{\sigma_2} := (\sigma_2(k_1), \dots, \sigma_2(k_{t'}), k_{t'+1}, \dots, k_d) \uparrow$  and  $(\beta^{\sigma_1})^{\sigma_2} := (l_1, \dots, l_{t'-1}, \sigma_2(l_{t'}), \dots, \sigma_2(l_d)) \uparrow$ . Then

$$p_{\alpha^{\sigma_1}} p_{\beta^{\sigma_1}} = \sum_{\sigma_2 \in [\alpha^{\sigma_1}, \beta^{\sigma_1}]} \pm p_{(\alpha^{\sigma_1})^{\sigma_2}} p_{(\beta^{\sigma_1})^{\sigma_2}}.$$

And this process may continue as there may be monomials  $p_{(\alpha^{\sigma_1})^{\sigma_2}} p_{(\beta^{\sigma_1})^{\sigma_2}}$  that are not standard and we must apply another shuffle. However this process will eventually terminate after a finite number of steps, guaranteed by the fact that there are only finitely many degree 2 monomials and the ordering property of the shuffles (cf. [LR08, Chapter 4]).

After substituting and combining like monomials we get that

$$(2.2.1) \quad p_\tau p_\phi = \sum_{\alpha, \beta} A_{\alpha, \beta} p_\alpha p_\beta \text{ with } A_{\alpha, \beta} \in \mathbb{C}, \alpha \geq \beta$$

where for each  $\alpha, \beta$  with  $A_{\alpha, \beta} \neq 0$  we have  $\alpha = (((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}$ ,  $\beta = (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}$  for some  $M > 0$  and some  $\sigma_M \in [(((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}, (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}]$ ,  $\dots, \sigma_2 \in [\alpha^{\sigma_1}, \beta^{\sigma_1}]$ ,  $\sigma_1 \in [\tau, \phi]$ . For a fixed  $\alpha, \beta$ , their description in terms of  $M$  and  $\sigma_1, \dots, \sigma_M$  may not be unique, as a particular standard monomial in the summation may be the result of multiple different chains of shuffles, which is why  $A_{\alpha, \beta}$  may equal integers other than  $-1, 0, 1$ .

In addition because of the ordering property of the shuffles, for each  $\alpha, \beta$  with  $A_{\alpha, \beta} \neq 0$  we have that  $\alpha = (((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M} > (((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}} > \dots > \alpha^{\sigma_1} > \tau$  and  $\beta = (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M} < (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}} < \dots < \beta^{\sigma_1} < \phi$ .

We refer to (2.2.1) as the result of the *degree 2 straightening process* applied on the nonstandard monomial  $p_\tau p_\phi$ .

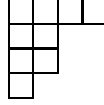
Finally a degree  $r$  nonstandard monomial may be straightened by inductively applying the degree 2 straightening algorithm. That process is the *degree  $r$  straightening process*.

To straighten a monomial on  $X(w)$  all that is required is to apply the straightening process for the Grassmannian and then to note that in the resulting sum of standard monomials, any that are standard but not standard on  $X(w)$  are equal to zero on  $X(w)$  (cf. Definition 2.1.4).

**2.3. Young Diagrams and Tableaux.** This section for the most part follows the terminology of [Ful97], [FH91], and [Sta99]. Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a collection of nonnegative integers with  $\lambda_1 \geq \dots \geq \lambda_k$ , then for  $|\lambda| := \lambda_1 + \dots + \lambda_k$  we say that  $\lambda$  is a *partition* of  $|\lambda|$ . We call the  $\lambda_i$  the *parts* of  $\lambda$ . It will be useful at times to make this notation more succinct by rewriting  $\lambda = (\lambda_1, \dots, \lambda_k)$ ,

replacing any maximal chain  $\lambda_i, \dots, \lambda_{i+j-1}$  where  $\lambda_i = \dots = \lambda_{i+j-1} = a$  with  $a^j$ . We identify a partition  $\lambda$  with its *Young diagram*, also denoted  $\lambda$  for simplicity of notation, which is a collection of left justified boxes with  $\lambda_i$  boxes in the  $i$ th row for  $1 \leq i \leq k$ . These boxes are referred to by specifying row and column, with the leftmost column denoted column 1, and the topmost row denoted row 1.

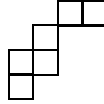
**Example 2.3.1.** The partition  $(4, 2, 2, 1) = (4, 2^2, 1)$  is identified with the Young diagram.



The *conjugate partition*  $\lambda'$  is the partition whose diagram is the transpose of the diagram of  $\lambda$ , or equivalently, it is defined by setting the part  $\lambda'_i := \#\{j \mid \lambda_j \geq i\}$ . The conjugate partition of  $(4, 2, 2, 1)$  from Example 2.3.1 is  $(4, 3, 1, 1)$ . A partition that has all parts equal to the same value is a *rectangle*, and one that has all parts equal to one of two values is a *fat hook*. A fat hook with all parts except the first equal to 1 is a *hook*. The partition  $(3, 3, 3, 3)$  is a rectangle, the partition  $(4, 4, 4, 2, 2)$  is a fat hook, and the partition  $(5, 1, 1)$  is a hook.

If we have a second partition  $\mu$  we write  $\mu \subseteq \lambda$  if the diagram for  $\mu$  is contained in the diagram for  $\lambda$ , or equivalently, if  $\mu_i \leq \lambda_i$  for  $i \geq 1$ . If  $\mu \subseteq \lambda$  we may define the skew diagram  $\lambda/\mu$  which is obtained by deleting the leftmost  $\mu_i$  boxes from row  $i$  of the diagram  $\lambda$  for each row of  $\lambda$ . The number of boxes in the skew diagram is equal to  $|\lambda/\mu| := |\lambda| - |\mu|$ . It is important to note here the fact that  $\lambda = \lambda/(0)$ , and so many definitions made for skew diagrams may be specialized to diagrams for partitions.

**Example 2.3.2.** The skew diagram for  $\lambda/\mu = (4, 2, 2, 1)/(2, 1)$  is



The  $\pi$ -rotation of a skew diagram  $\lambda/\mu$ , written  $(\lambda/\mu)^\pi$ , is obtained by rotating  $\lambda/\mu$  through  $\pi$  radians. Define  $\tilde{\lambda}/\tilde{\mu}$  to be the skew diagram obtained by deleting all empty rows and columns from the skew diagram  $\lambda/\mu$ . If  $\lambda \subseteq (m^n)$  for  $m, n$  positive integers, the  $(m^n)$ -complement of  $\lambda$  is denoted  $\lambda^*$ , with  $\lambda_k^* = m - \lambda_{n-k+1}$ . If  $\lambda \subseteq (m^n)$  for  $m, n$  positive integers, then  $(m^n)/\lambda$  is a skew diagram and  $((m^n)/\lambda)^\pi$  is always a partition and is equal to  $\lambda^*$ .

**Example 2.3.3.** The  $\pi$ -rotation of  $(4, 2, 2, 1)/(2, 1)$  is  $(5, 5, 3, 2)/(4, 2, 2)$ . Let  $m = 5$ ,  $n = 5$  then the partition  $(4, 2, 2, 1) \subseteq (m^n)$  and we have the  $(m^n)$ -complement of  $\lambda$  is  $\lambda^* = (5, 4, 3, 3, 1)$ . As noted above the  $\pi$ -rotation of  $(m^n)/(4, 2, 2, 1)$  is also  $(5, 4, 3, 3, 1)$ .

If  $\lambda \subseteq (m^n)$  for  $m, n$  positive integers, there is a unique shortest lattice path of length  $m + n$  dividing the boxes of  $\lambda$  and the boxes of  $(m^n)/\lambda$  starting at the bottom-left corner of the rectangle  $(m^n)$  and ending at the top-right corner of the rectangle. The  $m^n$ -shortness of  $\lambda$  is the length of the shortest line segment in this path.

**Example 2.3.4.** Let  $\lambda = (4, 2, 2, 1)$  and  $m = \lambda_1 = 4$ ,  $n = \lambda'_1 = 4$ . Then the lengths of the line segments for the lattice path are  $(1, 1, 1, 2, 2, 1)$  and so the  $4^4$ -shortness of  $\lambda$  is 1.

**Lemma 2.3.5.** Let  $\lambda, \mu$  be two partitions and let  $m, n, p$ , and  $q$  be positive integers such that  $\lambda \subseteq (m^n)$ ,  $\mu \subseteq (p^q)$  and the skew diagrams  $(m^n)/\lambda$  and  $(p^q)/\mu$  have no empty rows or columns. If  $((m^n)/\lambda)^\pi = ((p^q)/\mu)^\pi$  then  $m = p$ ,  $n = q$ , and  $\lambda = \mu$ .



*Proof.* Set  $\gamma = ((m^n)/\lambda)^\pi$  and  $\nu = ((p^q)/\mu)^\pi$ . Then  $\gamma, \nu$  are partitions. We have  $\gamma_k = m - \lambda_{n-k+1}$  for  $1 \leq k \leq n$  and  $\gamma_k = 0$  for  $k > n$ . Similarly  $\nu_k = p - \mu_{q-k+1}$  for  $1 \leq k \leq q$  and  $\nu_k = 0$  for  $k > q$ . By assumption we have that  $\gamma = \nu$  so  $m - \lambda_{n-k+1} = p - \mu_{q-k+1}$  for  $k \geq 1$ .

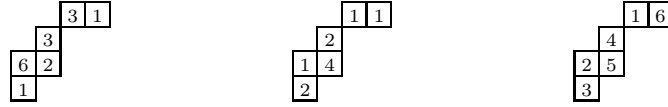
Suppose  $n \neq q$ . Without loss of generality assume that  $q > n$ . Then  $\gamma_q = 0$ , which implies that  $\nu_q = 0$ , that is that  $p - \mu_1 = 0$ . But this would mean that  $p = \mu_1$  which would indicate that  $(p^q)/\mu$  had an empty first row. This is a contradiction of our initial assumption and thus  $n = q$ .

Now we know that  $\lambda_n = 0$  because  $(m^n)/\lambda$  has no empty columns. We also know that  $\mu_n = \mu_q = 0$  because  $(p^q)/\mu$  has no empty columns. But then  $\gamma_1 = \nu_1$  implies that  $m - \lambda_n = p - \mu_n$ . Thus  $m = p$ .

Now the fact that  $m = p$ ,  $n = q$  trivially implies that  $\lambda = \mu$ .  $\square$

A tableaux on  $\lambda/\mu$  is an assignment of a positive integer to each box of  $\lambda/\mu$ . A semistandard (Young) tableaux, often abbreviated SSYT, is a tableaux where the values in each box increase weakly along each row, and increase strictly down each column. A standard (Young) tableaux is a semistandard tableaux where the values in each box increase strictly along each row.

**Example 2.3.6.** A tableaux, semistandard tableaux, and standard tableaux on  $(4, 2, 2, 1)/(2, 1)$ .



If we fix a partition  $\lambda$  and a bound  $M$  on the maximum value that can be assigned to a box in a tableaux  $T$  we may define the schur function  $s_\lambda$  by

$$s_\lambda = \sum_{T \text{ a SSYT on } \lambda} x_1^{\# \text{ of 1's in } T} \cdots x_M^{\# \text{ of } M\text{'s in } T}.$$

In the same way for a skew diagram  $\lambda/\mu$  we may define the skew schur function  $s_{\lambda/\mu}$ . Both the schur functions and the skew schur functions are symmetric functions, and the schur functions form a vector space basis of the ring of symmetric functions in the variables  $x_1, \dots, x_M$ . Thus the product of two schur functions, which is itself a symmetric function, can be written as a sum of schur functions

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda, \mu}^{\nu} s_{\nu}$$

and this is one of many equivalent ways of defining the Littlewood-Richardson coefficients  $c_{\lambda, \mu}^{\nu}$ . Note that the above sum is over all partitions  $\nu$  such that  $|\nu| = |\lambda| + |\mu|$ .

The Littlewood-Richardson coefficients are also critical in describing the expansion of the skew schur functions  $s_{\lambda/\mu}$  in terms of the schur functions, namely

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu, \nu}^{\lambda} s_{\nu}$$

and the above sum is over all partitions  $\nu$  such that  $|\nu| = |\lambda| - |\mu|$ . In the special case where for a fixed skew schur function  $s_{\lambda/\mu}$  all the  $c_{\mu, \nu}^{\lambda}$  are either 0 or 1 we say that  $s_{\lambda/\mu}$  is multiplicity-free. The reason for this designation will become clear in Section 2.4.

The study of the Littlewood-Richardson coefficients is a rich subject with applications in the decomposition of tensor products of  $GL_N$  representations in characteristic zero, intersection numbers on the Grassmannian, and the eigenvalues of sums of Hermitian matrices (cf. [FH91], [HL12], [TY10], and [Ful00]).

We will need a few standard identities whose derivations can be found in [Sta99]. The first are two non-trivial symmetries of the Littlewood-Richardson coefficients with regard to the partitions;

$$(2.3.7) \quad c_{\mu, \nu}^{\lambda} = c_{\nu, \mu}^{\lambda} \text{ and } c_{\mu, \nu}^{\lambda} = c_{\mu', \nu'}^{\lambda'}.$$

The second are two important identities for skew schur functions,

$$(2.3.8) \quad s_{\lambda/\mu} = s_{\tilde{\lambda}/\tilde{\mu}}$$

and

$$(2.3.9) \quad s_{\lambda/\mu} = s_{(\lambda/\mu)^\pi}$$

which state that deletion of empty rows and columns of a skew diagram as well as  $\pi$ -rotation do not affect the associated skew schur function. Any skew schur function whose associated skew diagram has no empty rows or columns is called *basic*, and so (2.3.8) implies that any skew schur function is equivalent to a basic skew schur function.

The final result we will need, due to C. Gutschwager[Gut10], and to Thomas and Yong[TY10], is a characterization of all multiplicity-free basic skew schur functions.

**Theorem 2.3.10.** *The basic skew schur function  $s_{\lambda/\mu}$  is multiplicity-free if and only if  $\lambda$  and  $\mu$  satisfy one or more of the following conditions:*

- (a)  $\mu$  or  $\lambda^*$  is the zero partition
- (b)  $\mu$  or  $\lambda^*$  is a rectangle of  $m^n$ -shortness 1
- (c)  $\mu$  is a rectangle of  $m^n$ -shortness 2 and  $\lambda^*$  is a fat hook (or vice versa)
- (d)  $\mu$  is a rectangle and  $\lambda^*$  is a fat hook of  $m^n$ -shortness 1 (or vice versa)
- (e)  $\mu$  and  $\lambda^*$  are rectangles

where  $m = \lambda_1$ ,  $n = \lambda'_1$ , and  $\lambda^*$  is the  $m^n$ -complement of  $\lambda$ .

**2.4. Schur and Weyl Modules.** The skew Schur functions  $s_{\lambda/\mu}$  are the characters of certain representations of  $GL_n$  for some  $n \geq 1$ , where  $n$  is the bound on the entries in the boxes(cf. [FH91]). Given a standard tableaux on the skew diagram  $\lambda/\mu$  with the bound on the entries in the boxes equal to  $d$  we may define two subgroups of the symmetric group  $S_d$

$$Row^{\lambda/\mu} := \{\sigma \in S_d \mid \sigma \text{ permutes the enties in each row among themselves}\}$$

and

$$Col^{\lambda/\mu} := \{\sigma \in S_d \mid \sigma \text{ permutes the entries in each column among themselves}\}.$$

In the group algebra  $\mathbb{C}[S_d]$  we introduce two elements, called the Young symmetrizers

$$\Upsilon_W^{\lambda/\mu} := \sum_{\substack{\sigma \in Row^{\lambda/\mu} \\ \rho \in Col^{\lambda/\mu}}} sign(\rho)\sigma\rho$$

and

$$\Upsilon_S^{\lambda/\mu} := \sum_{\substack{\sigma \in Row^{\lambda/\mu} \\ \rho \in Col^{\lambda/\mu}}} sign(\rho)\rho\sigma.$$

Let  $V = \mathbb{C}^n$  with standard basis  $\{e_1, \dots, e_n\}$ . The symmetric group  $S_d$  acts on the  $d$ th tensor product  $V^{\otimes d}$  on the right by permuting the factors, while  $GL_n$  acts on  $V$  on the left and thus diagonally on  $V^{\otimes d}$  on the left. The fact that this left action of  $GL_n$  commutes with the right action of  $S_d$  is the source of Schur-Weyl duality and gives the relationship between the irreducible finite-dimensional representations of the general linear and symmetric groups.

The Schur Module  $\mathbb{S}^{\lambda/\mu}(V)$  and Weyl Module  $\mathbb{W}^{\lambda/\mu}(V)$  are defined to be

$$\mathbb{S}^{\lambda/\mu}(V) := (V^{\otimes d})\Upsilon_S^{\lambda/\mu}$$

and

$$\mathbb{W}^{\lambda/\mu}(V) := (V^{\otimes d})\Upsilon_W^{\lambda/\mu}.$$

These are  $\mathrm{GL}_n$  representations spanned by all the young symmetrized tensors in  $V^{\otimes d}$ . In characteristic zero these representations are related by the identity  $\mathbb{S}^{\lambda/\mu}(V) \cong \mathbb{W}^{\lambda'/\mu'}(V)$  [Wey03, Proposition 2.1.18(c)] and for our purposes it will prove to be convenient to focus on the Weyl Modules  $\mathbb{W}^{\lambda/\mu}(V)$ .

Given a tableaux  $T$  of  $\lambda/\mu$  numbered with  $\{1, \dots, n\}$  we can associate to  $T$  a decomposable tensor

$$e_T = \bigotimes_{i=1}^{\lambda_1} e_{T(-,i)}$$

where  $e_{T(-,i)}$  is the tensor product, in order, of those basis vectors whose indices appear in column  $i$ . Multilinearity implies that  $\mathbb{W}^{\lambda/\mu}(V)$  is spanned by  $e_T \Upsilon_W^{\lambda/\mu}$  as  $T$  ranges over all tableau. In fact we may do better, as the following theorem illustrates.

**Theorem 2.4.1.** *The set  $\{e_T \mid T \text{ is a semistandard tableaux on } \lambda/\mu\}$  is a  $\mathbb{C}$ -basis for  $\mathbb{W}^{\lambda/\mu}(V)$ .*

*Proof.* This is a well known result, and a sketch of the details may be found in [FH91, Exercise 6.15 and 6.19]  $\square$

Note that the above construction works for any Young diagram  $\lambda$  merely by noting that  $\lambda$  has the same diagram as the skew diagram  $\lambda/(0)$ . In fact the Weyl Module  $\mathbb{W}^\lambda(V) := \mathbb{W}^{\lambda/(0)}(V)$  is an irreducible  $\mathrm{GL}_n$  representation and any Weyl Module  $\mathbb{W}^{\lambda/\mu}(V)$  can be written as a direct sum of these irreducible modules by

$$(2.4.2) \quad \mathbb{W}^{\lambda/\mu}(V) \cong \bigoplus_{\nu} \mathbb{W}^{\nu}(V)^{\oplus c_{\mu,\nu}^{\lambda}}$$

where the  $c_{\lambda\mu}^{\nu}$  are the Littlewood-Richardson coefficients defined in Section 2.3, and the direct sum is over all partitions  $\nu$  such that  $|\nu| = |\lambda| - |\mu|$ .

**Remark 2.4.3.** The above decomposition implies that  $\mathbb{W}^{\lambda/\mu}(V)$  will be multiplicity free when the skew Schur function  $s_{\lambda/\mu}$  is multiplicity free, and the skew diagrams for which this occurs are enumerated by Theorem 2.3.10.

**Definition 2.4.4.** Let  $r$  be a positive integer. Define the  $\mathrm{GL}_M$  representation  $\det_M^r : \mathrm{GL}_M \rightarrow \mathbb{C}^*$ ,  $\det_M^r(g) = (\det(g))^r$ ,  $g \in \mathrm{GL}_M$ . Then  $\det_M^{-r}$  is defined to be the dual of  $\det_M^r$ .

We have the following isomorphism of  $\mathrm{GL}_M$ -modules

$$(2.4.5) \quad (\mathbb{W}^\lambda(\mathbb{C}^M))^* \otimes \det_M^r \cong \mathbb{W}^{(r^M)/\lambda}(\mathbb{C}^M) \cong \mathbb{W}^{((r^M)/\lambda)^\pi}(\mathbb{C}^M).$$

A proof of the first isomorphism may be found in [Mag98, Theorem 6(c)] although the notation used is different from ours, in particular our  $\mathbb{W}^\lambda$  is denoted  $S_\lambda$ . The second isomorphism follows from (2.3.9).

**2.5. The Littlewood-Richardson Rule.** We will need to be able to calculate the value of certain Littlewood-Richardson coefficients in Section 4. To do this we recall the Littlewood-Richardson Rule from [Ful97, Section 5].

Given a semistandard tableaux or a semistandard skew tableaux  $T$  we define the *row word*, denoted  $w_{row}(T)$ , of the tableaux  $T$  to be the entries of  $T$  read from left to right and bottom to top. A row word  $w_{row}(T) = x_1, \dots, x_r$  is called a reverse lattice word if in every reversed sequence  $x_r, x_{r-1}, \dots, x_{s+1}, x_s$  the number  $i$  appears at least as often as  $i+1$  for all  $i$  and all  $1 \leq s < r$ .

**Example 2.5.1.** Consider the following two skew tableaux



The associated row words are 2,1,3,2,1,1 and 2,1,3,2,2,1. The first is a reverse lattice word and the second is not because 1,2,2 has more 2's than 1's.

A semistandard skew tableaux  $T$  is called a semistandard Littlewood-Richardson skew tableaux if  $w_{row}(T)$  is a reverse lattice word. A semistandard skew tableaux has weight  $\nu = (\nu_1, \dots, \nu_m)$  if the tableaux has  $\nu_1$  1's,  $\nu_2$  2's, ..., and  $\nu_m$  m's.

With these definitions we may state the Littlewood-Richardson Rule.

**Proposition 2.5.2.** ([Ful97, Proposition 5.3]) *The Littlewood-Richardson coefficient  $c_{\mu,\nu}^\lambda$  is equal to the number of semistandard Littlewood-Richardson skew tableaux on the shape  $\lambda/\mu$  with weight  $\nu$ .*

**Lemma 2.5.3.** *Let  $n \geq 1$ .*

- (a) *Let  $\lambda = (3^n, 2)$ ,  $\mu = (1)$ , and  $\nu = (3^n, 1)$ . Then  $c_{\mu,\nu}^\lambda = 1$ .*
- (b) *Let  $\lambda = (3^n, 1, 1)$ ,  $\mu = (1)$ , and  $\nu = (3^n, 1)$ . Then  $c_{\mu,\nu}^\lambda = 1$ .*
- (c) *Let  $\lambda = (3^n, 2)$ ,  $\mu = (2)$ , and  $\nu = (3^{n-1}, 2, 1)$ . Then  $c_{\mu,\nu}^\lambda = 1$ .*
- (d) *Let  $\lambda = (3^n, 2)$ ,  $\mu = (1, 1)$ , and  $\nu = (3^{n-1}, 2, 1)$ . Then  $c_{\mu,\nu}^\lambda = 1$ .*

*Proof.* Note that for (c) and (d) when  $n = 1$  we consider  $\nu = (3^0, 2, 1) = (2, 1)$ .

(a): By (2.3.7) we have that  $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$ . So according to Proposition 2.5.2 to find  $c_{\nu,\mu}^\lambda$  we must calculate the number of semistandard Littlewood-Richardson skew tableaux on the shape  $\lambda/\nu$  with weight  $\mu$ . But  $\lambda/\nu = (3^n, 2)/(3^n, 1)$  is a single box and there is clearly only one semistandard Littlewood-Richardson skew tableaux with weight  $\mu = (1)$ , that is with a single 1. Thus  $c_{\nu,\mu}^\lambda = 1$ .

(b): This proceeds in exactly the same way as part (a).

(c): By (2.3.7) we have that  $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda$ . We have that  $\lambda/\nu = (3^n, 2)/(3^{n-1}, 2, 1)$  is two disconnected boxes, and there is only one semistandard Littlewood-Richardson skew tableaux with weight  $\mu = (2)$ , it is the tableaux with a 1 in each box. Thus by Proposition 2.5.2 we have  $c_{\nu,\mu}^\lambda = 1$ .

(d): Similarly to part (c) we need to calculate the number of semistandard Littlewood-Richardson skew tableaux of shape  $(3^n, 2)/(3^{n-1}, 2, 1)$ , but in this case with weight  $\mu = (1, 1)$ . There are two possible fillings with this weight, however only the one with a 1 in the upper right box and a 2 in the lower left box has a row word that is a reverse lattice word. Thus  $c_{\mu,\nu}^\lambda = c_{\nu,\mu}^\lambda = 1$ .  $\square$

### 3. DECOMPOSITION RESULTS

**3.1. Blocks, Heads, and Partitions in Degree 1.** Let  $P = P_{\hat{d}}$  and  $w := (i_1, \dots, i_d) \in W^P$ . Then  $X(w)$  is a Schubert variety in  $G_{d,N}$ . Let  $Q_w$  be the stabilizer of  $X(w)$  in  $GL_N$  for the action of left multiplication. Throughout this paper when we discuss the stabilizers of Schubert varieties in  $GL_N$  it will always be for the action of left multiplication.

**Proposition 3.1.1.** *Define*

$$R'_{Q_w} := \{ n \in \{1, \dots, N-1\} \mid \exists m \text{ with } n = i_m \text{ and } i_m + 1 \neq i_{m+1} \}.$$

*Then  $Q_w = P_{R_{Q_w}}$  where  $R_{Q_w} = \{1, \dots, N-1\} \setminus R'_{Q_w}$ .*

*Proof.* This is immediate from the fact that  $R_{Q_w} = \{m \in \{1, \dots, N-1\} \mid s_{\alpha_m} w \leq w\}$  [LMS74].  $\square$

Let  $Q$  be a parabolic subgroup of  $GL_N$  that is a subgroup of  $Q_w$ , then we have  $Q = P_{R_Q}$  for some  $R_Q \subseteq R_{Q_w}$  (cf. Remark 2.1.2). Our main group of interest will be the reductive group  $L$ , defined as the Levi part of  $Q$ , and its Lie algebra  $\mathfrak{l} = \text{Lie}(L)$ . The group  $L$  acts on  $X(w)$  by left multiplication and this induces an action on the coordinate ring  $\mathbb{C}[X(w)]$ . This in turn induces an action of  $\mathfrak{l} := \text{Lie}(L)$  on  $\mathbb{C}[X(w)]$ . We explore this induced action in greater depth in Section 3.3.

The ultimate goal of this section is to describe a decomposition of the  $L$ -module  $\mathbb{C}[X(w)]$  into irreducible  $L$ -modules for a general  $w \in W^P$ . This is achieved in Theorem 3.5.3 and Corollary 3.5.9.

We start by introducing some notation. Define  $H_w := \{\tau \in W^P \mid \tau \leq w\}$ . Let  $\mathcal{H}_w$  be the Hasse diagram for the Bruhat order on  $H_w$ . We may label the edges of  $\mathcal{H}_w$  in the following way. Given an edge  $e$  connecting  $\tau_1$  to  $\tau_2$  with  $\tau_1 \leq \tau_2$  we know that  $\tau_1 = s_\beta \tau_2$  for a unique  $\beta \in R^+$ . However, in the case of the Grassmannian, we know that  $\beta$  is a simple root. This is because a divisor of  $X(\tau)$ , for  $\tau = (j_1, \dots, j_d)$ , is obtained by reducing a single entry, say  $j_m$ , to  $j_m - 1$ , in which case  $\beta$  is simply  $\alpha_{j_m-1}$ . Thus we may label the edge  $e$  by the unique  $s_{\alpha_r}$  such that  $\tau_1 = s_{\alpha_r} \tau_2$ ,  $\alpha_r \in S$ .

Let  $R'_Q := \{1, \dots, N-1\} \setminus R_Q$ , and set  $d'_L = |R'_Q| + 1$ . Then  $R'_Q$  can be written uniquely as the ascending sequence  $(a_1, \dots, a_{d'_L-1})$ . Define the augmented sequence  $\hat{a} := (\hat{a}_1, \dots, \hat{a}_{d'_L+1}) := (0, a_1, \dots, a_{d'_L-1}, N)$ .

**Definition 3.1.2.** Using the augmented sequence  $\hat{a}$  we may partition  $\{1, \dots, N\}$  into subsets  $\text{Block}_{L,k} := \{\hat{a}_k + 1, \dots, \hat{a}_{k+1}\}$  for  $1 \leq k \leq d'_L$ . We will refer to these as the *blocks* of  $L$ . Let  $N_k := |\text{Block}_{L,k}| = \hat{a}_{k+1} - \hat{a}_k$ . Thus  $d'_L$  is the number of blocks of  $L$ .

**Remark 3.1.3.** These blocks are closely related to the form of  $L$  and  $\mathfrak{l}$ . In particular,  $L = GL_{N_1} \times \dots \times GL_{N_{d'_L}}$  and  $\mathfrak{l} = \mathfrak{gl}_{N_1} \times \dots \times \mathfrak{gl}_{N_{d'_L}}$ . Thus our decomposition of the  $L$ -module  $\mathbb{C}[X(w)]$  into irreducible  $L$ -modules will be in terms of tensor products of Weyl modules associated to the  $GL_{N_i}$ .

**Definition 3.1.4.** Given an element  $\tau = (j_1, \dots, j_d) \in W^P$ , we define the *class* of  $\tau$  as the sequence  $\text{Class}_\tau := (c_1, \dots, c_d)$  where each  $c_m$  is equal to the unique  $k$  such that  $j_m \in \text{Block}_{L,k}$ .

**Proposition 3.1.5.** Let  $\tau = (j_1, \dots, j_d) \in H_w$ . The following properties of  $\tau$  are equivalent.

- (i) The subvariety  $X(\tau)$  is  $L$ -stable.
- (ii) The  $\text{wt}(\mathfrak{p}_\tau)$  is  $L$ -dominant.
- (iii) For all  $1 \leq i \leq d'_L$  we have  $\tau \cap \text{Block}_{L,k}$  is either empty or contains a maximal collection of elements from  $\text{Block}_{L,k}$ , explicitly for all  $m \in \tau \cap \text{Block}_{L,k}$  and all  $n \in \text{Block}_{L,k} \setminus \{\tau \cap \text{Block}_{L,k}\}$  we have  $m > n$ .

*Proof.* Let  $Q_\tau$  be the stabilizer of  $X(\tau)$ . The  $L$ -stability of  $X(\tau)$  is equivalent to  $Q \subseteq Q_\tau$ , or that  $R_Q \subseteq R_{Q_\tau}$ .

(iii)  $\Rightarrow$  (i): Suppose (iii) holds for  $\tau$ . We have  $Q_\tau = P_{R_{Q_\tau}}$  where

$$R_{Q_\tau} = \{m \in \{1, \dots, N-1\} \mid s_{\alpha_m} \tau \leq \tau\}.$$

Suppose  $m \in R'_{Q_\tau}$ , then  $s_{\alpha_m} \tau > \tau$ . The description of  $\tau$  given in (iii) implies that  $m = \hat{a}_k$  for some  $k = 2, \dots, d'_L$ . Thus  $m \in R'_Q$ , which implies  $R'_{Q_\tau} \subseteq R'_Q$ . It follows that  $R_Q \subseteq R_{Q_\tau}$ .

(i)  $\Rightarrow$  (iii): Suppose (iii) does not hold for  $\tau$ . This implies there is a block, say  $\text{Block}_{L,k}$ , such that there is a  $m \in \tau \cap \text{Block}_{L,k}$  with  $m+1 \in \text{Block}_{L,k} \setminus \{\tau \cap \text{Block}_{L,k}\}$ . Then  $m \in \tau$  and  $m+1 \notin \tau$ . This implies that  $s_{\alpha_m} \tau > \tau$  and thus  $m \notin R_{Q_\tau}$ . Since  $m$  and  $m+1$  are both in  $\text{Block}_{L,k}$  we have that  $m$  is not the maximal element in the block and thus  $m \in R_Q$ . Thus  $R_Q \not\subseteq R_{Q_\tau}$  and hence  $Q \not\subseteq Q_\tau$ .

(iii)  $\Rightarrow$  (ii): As discussed in Remark 2.1.3 we have that the weight of  $\mathfrak{p}_\tau$  is given in the  $\epsilon$ -basis by the sequence  $\chi_\tau := (\chi_1, \dots, \chi_N)$  where

$$\chi_m := \begin{cases} 0 & m \in \tau \\ 1 & m \notin \tau \end{cases}$$

for all  $1 \leq m \leq N$ .

Then by Remark 3.1.3 and Remark 2.1.1 we have that  $\chi_\tau$  is  $L$ -dominant if and only if when we partition the sequence  $(\chi_1, \dots, \chi_N)$  into subsequences  $\chi_\tau^{(k)} := (\chi_{\hat{a}_k+1}, \dots, \chi_{\hat{a}_{k+1}})$  for  $1 \leq k \leq d'_L$ , each sequence  $\chi_\tau^{(k)}$  is non-increasing.

Since (iii) holds for  $\tau$  we have that each  $\chi_\tau^{(k)}$  is of the form  $(1, \dots, 1, 0, \dots, 0)$  which is non-increasing. Thus  $\chi_\tau$  is  $L$ -dominant.

(ii)  $\Rightarrow$  (iii): Suppose (iii) does not hold for  $\tau$ . Then there is a block, say  $\text{Block}_{L,k}$ , such that there is a  $m \in \tau \cap \text{Block}_{L,k}$  with  $m+1 \in \text{Block}_{L,k} \setminus \{\tau \cap \text{Block}_{L,k}\}$ . But then  $\chi_\tau^{(k)}$  is not non-increasing since  $m, m+1 \in \{\hat{a}_k+1, \dots, \hat{a}_{k+1}\}$  and  $\chi_m = 0$  and  $\chi_{m+1} = 1$ . □

**Definition 3.1.6.** Let  $\tau \in H_w$ . If any of the three equivalent properties from Proposition 3.1.5 hold for  $\tau$  we call  $\tau$  a *head of type  $L$* . And we define

$$\text{Head}_L := \{\tau \in H_w \mid \tau \text{ is a head of type } L\}.$$

**Example 3.1.7.** Let  $d = 3$  and  $N = 9$ . Consider  $w = (3, 6, 9) \in W^{\hat{P}_3}$ . Then  $X(w)$  is a Schubert variety in  $G_{3,9}$ . In this case  $R'_{Q_w} = \{3, 6\}$  and  $R_{Q_w} = \{1, 2, 4, 5, 7, 8\}$ . Choose  $R_Q := R_{Q_w}$  for the parabolic subgroup  $Q = P_{R_Q}$ . Then  $\hat{a} = \{0, 3, 6, 9\}$ . So  $\text{Block}_{L,1} = (1, 2, 3)$ ,  $\text{Block}_{L,2} = (4, 5, 6)$ , and  $\text{Block}_{L,3} = (7, 8, 9)$ . Then

$$\text{Head}_L = \{(1, 2, 3), (2, 3, 6), (2, 3, 9), (3, 5, 6), (3, 6, 9)\}.$$

The head  $(2, 3, 6)$  has  $\text{Class}_{(2,3,6)} = (1, 1, 2)$ .

We now prove a handful of technical lemmas relating to heads of type  $L$ , blocks of  $L$ , and classes that will prove useful throughout this section. Our first goal will be to show that given a  $w \in W^P$  we may describe a particular partition of the Hasse diagram into disjoint subdiagrams. This partition will turn out to influence the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L$ -modules.

**Lemma 3.1.8.** Let  $\theta_1, \theta_2 \in \text{Head}_L$ . Then  $\theta_1 = \theta_2$  if and only if  $\text{Class}_{\theta_1} = \text{Class}_{\theta_2}$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $\text{Class}_{\theta_1} = \text{Class}_{\theta_2}$ . This would imply that  $|\theta_1 \cap \text{Block}_{L,k}| = |\theta_2 \cap \text{Block}_{L,k}|$  for all  $1 \leq j \leq d'_L$ . Since each of these is a head,  $\theta_i \cap \text{Block}_{L,k}$  for  $i = 1, 2$  must be a maximal collection of elements from  $\text{Block}_{L,k}$ . These two results imply that  $\theta_1 \cap \text{Block}_{L,k} = \theta_2 \cap \text{Block}_{L,k}$  for all  $1 \leq k \leq d'_L$ . But this, combined with the fact that the blocks of type  $L$  partition  $\{1, \dots, N\}$ , implies that  $\theta_1 = \theta_2$ .

( $\Leftarrow$ ) Suppose  $\text{Class}_{\theta_1} \neq \text{Class}_{\theta_2}$ . Then for some block, say  $\text{Block}_{L,k}$ , we have  $|\theta_1 \cap \text{Block}_{L,k}| \neq |\theta_2 \cap \text{Block}_{L,k}|$ . But this implies  $\theta_1 \neq \theta_2$ . □

**Lemma 3.1.9.** Let  $m \in R_Q$  and  $\tau \in H_w$ . Then  $\text{Class}_\tau = \text{Class}_{s_{\alpha_m} \tau}$ .

*Proof.* Let  $\tau = (j_1, \dots, j_d) \in H_w$ . We have  $m \in \text{Block}_{L,k}$  for some  $1 \leq k \leq d'_L$ . Suppose  $m+1 \notin \text{Block}_{L,k}$ . This would mean that  $m$  is the maximal element in  $\text{Block}_{L,k}$ , so  $m = \hat{a}_{k+1}$ . Thus  $m \in R'_Q$  or  $m = N$ . In either case this means  $m \notin R_Q$ . This is a contradiction and thus  $m+1 \in \text{Block}_{L,k}$ . Now  $s_{\alpha_m}$  acts on  $\tau$  in one of the following ways.

**Case 1:**  $\exists n$  such that  $j_n = m$ , with  $j_{n+1} \neq m+1$ . Then  $s_{\alpha_m} \tau = (j_1, \dots, j_{n-1}, m+1, j_{n+1}, \dots, j_d)$ .

**Case 2:**  $\exists n$  such that  $j_n = m+1$ , with  $j_{n-1} \neq m$ . Then  $s_{\alpha_m} \tau = (j_1, \dots, j_{n-1}, m, j_{n+1}, \dots, \tau_d)$ .

**Case 3:**  $\exists n$  such that  $j_n = m$ , with  $j_{n+1} = m+1$ . Then  $s_{\alpha_m} \tau = \tau$ .

**Case 4:**  $\nexists n$  such that  $\tau_n = m$  or  $\tau_n = m+1$ . Then  $s_{\alpha_m} \tau = \tau$ .

Thus in all four possible cases it can be seen that  $\text{Class}_\tau = \text{Class}_{s_{\alpha_m} \tau}$ . □

**Lemma 3.1.10.** Let  $\tau := (j_1, \dots, j_d) \in H_w$ . Then  $\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \theta$  for some  $m_1, \dots, m_t \in R_Q$  and  $\theta \in \text{Head}_L$ .



*Proof.* To see this fix a  $k$  with  $1 \leq k \leq d'_L$  and consider  $\text{Block}_{L,k}$ . If  $\tau \cap \text{Block}_{L,k}$  is empty we are done with this block. Otherwise  $\tau \cap \text{Block}_{L,k} = (j_m, j_{m+1}, \dots, j_{m+r})$  for some  $1 \leq m, r \leq d$  with  $m+r \leq d$ . If this is a maximal collection of elements in  $\text{Block}_{L,k}$  we are done with this block, otherwise we may make it maximal. We have

$$s_{\alpha_{\hat{a}_{k+1}-1}} \cdots s_{\alpha_{j_{m+r+1}}} s_{\alpha_{j_{m+r}}} (j_m, \dots, j_{m+r}) = (j_m, \dots, j_{m+r-1}, \hat{a}_{k+1}).$$

It is clear that each  $\hat{a}_{k+1} - 1, \dots, m+r+1, m+r \in R_Q$ . Each time we act, it matches case 1 from Lemma 3.1.9, it increments only the largest value in the block.

And now we may do the same to the second largest element in the block, incrementing it to  $\hat{a}_{k+1} - 1$ , giving

$$s_{\alpha_{\hat{a}_{k+1}-2}} \cdots s_{\alpha_{j_{m+r}}} s_{\alpha_{j_{m+r-1}}} (j_m, \dots, j_{m+r-1}, \hat{a}_{k+1}) = (j_m, \dots, j_{m+r-2}, \hat{a}_{k+1} - 1, \hat{a}_{k+1}).$$

By induction on this process we have

$$s_{\alpha_{\hat{a}_{k+1}-r}} \cdots s_{\alpha_{j_m}} \cdots s_{\alpha_{\hat{a}_{k+1}-1}} \cdots s_{\alpha_{j_{m+r}}} (j_m, \dots, j_{m+r}) = (\hat{a}_{k+1} - r, \dots, \hat{a}_{k+1})$$

which is maximal in  $\text{Block}_{L,k}$ . Note that all the above simple reflections are associated to  $\alpha_n$  for  $n \in \text{Block}_{L,k}$  and  $n \in R_Q$ . Thus these only affect the entries of  $\tau$  that intersect with  $\text{Block}_{L,k}$  and can be performed independently for each block.

And thus after performing the incrementing process for each block we have  $s_{\alpha_{m_t}} \cdots s_{\alpha_{m_1}} \tau$  is a head of type  $L$  for  $t \in \mathbb{N}$ ,  $m_1, \dots, m_t \in R_Q$ . That is  $s_{\alpha_{m_t}} \cdots s_{\alpha_{m_1}} \tau = \theta$  for some  $\theta \in \text{Head}_L$ . Rewriting we get our desired result  $\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \theta$ .  $\square$

The following important proposition is now almost trivial with these lemmas.

**Proposition 3.1.11.** (*Partition in Degree 1*) Let  $\hat{\mathcal{H}}_w$  be the diagram formed by removing all edges of  $\mathcal{H}_w$  labeled by  $s_{\alpha_m}$  with  $m \in R'_Q$ . Then  $\hat{\mathcal{H}}_w$  is a disconnected diagram with  $|\text{Head}_L|$  disjoint subdiagrams and each subdiagram has a unique maximal element under the Bruhat order given by a  $\theta \in \text{Head}_L$ . Further the class of each element in a fixed subdiagram is equal to the class of the head in that subdiagram.

*Proof.* By Lemma 3.1.10 we have that for any  $\tau \in H_w$ ,  $\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \theta$  for some  $t \in \mathbb{N}$ ,  $m_1, \dots, m_t \in R_Q$  and  $\theta \in \text{Head}_L$ . Thus there is a path of edges in  $\hat{\mathcal{H}}_w$  connecting  $\tau$  and  $\theta$ . By Lemma 3.1.9 this also means  $\text{Class}_\tau = \text{Class}_\theta$ . Combining this with Lemma 3.1.8 implies that  $\theta$  is in fact the unique head connected to  $\tau$  in  $\hat{\mathcal{H}}_w$ . And thus removing all edges of  $\mathcal{H}_w$  labeled by  $s_{\alpha_m}$  with  $m \in R'_Q$  to form  $\hat{\mathcal{H}}_w$  leaves a disconnected diagram with  $|\text{Head}_L|$  disjoint subdiagrams.

It remains to show that the unique maximal element in each subdiagram is in fact the head. But this is clear by the proof of Lemma 3.1.10. For every  $\tau \in H_w$  we found  $s_{\alpha_{m_t}} \cdots s_{\alpha_{m_1}} \tau = \theta$  for some  $t \in \mathbb{N}$ ,  $m_1, \dots, m_t \in R_Q$  and  $\theta \in \text{Head}_L$ . And each of these  $s_{\alpha_{m_n}}$  for  $1 \leq n \leq t$  acted by increasing the Bruhat order. And thus for all  $\tau$  that are connected to  $\theta$  in  $\hat{\mathcal{H}}_w$  we have  $\theta \geq \tau$ .  $\square$

**Definition 3.1.12.** Let  $\tau \in H_w$ . Then define the head of  $\tau$ , which we will denote  $\theta_\tau^{L,w}$ , to be the unique head in  $\text{Head}_L$  connected to  $\tau$  in  $\hat{\mathcal{H}}_w$ . In general, we will write  $\theta_\tau^{L,w}$  as  $\theta_\tau$ , whenever no confusion will arise from the omission.

The uniqueness of the head  $\theta_\tau$  of a  $\tau \in H_w$  is implied by Proposition 3.1.11.

**Definition 3.1.13.** For  $\theta \in \text{Head}_L$  define

$$\text{WStd}_\theta := \{\tau \in H_w \mid \theta_\tau = \theta\}.$$

Thus  $\text{WStd}_\theta$  is the collection of all elements connected to  $\theta$  in the disjoint Hasse diagram  $\hat{\mathcal{H}}_w$ .

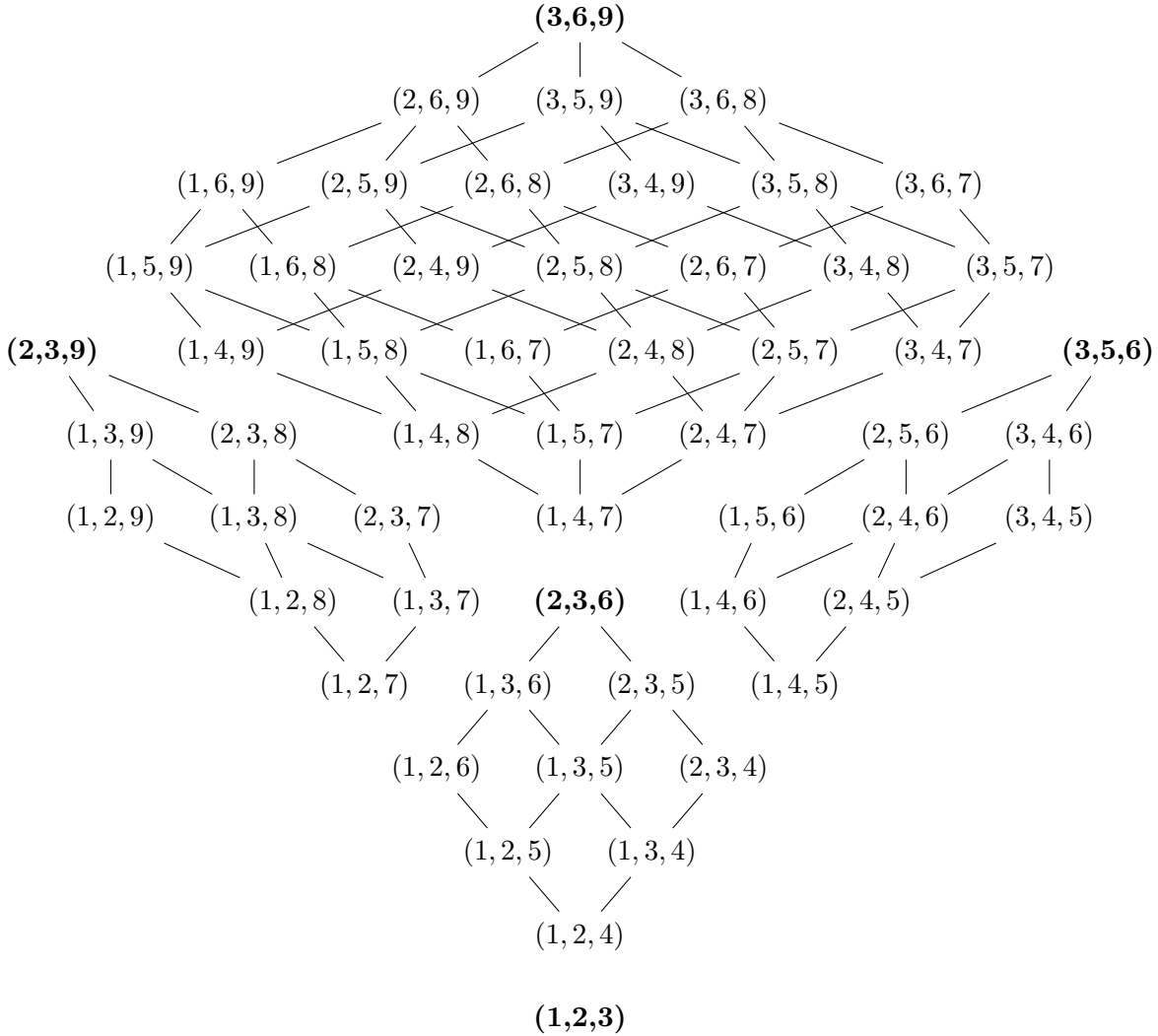
**Corollary 3.1.14.** *Let  $\tau_1, \tau_2 \in H_w$ . Then  $\theta_{\tau_1} = \theta_{\tau_2}$  if and only if  $\text{Class}_{\tau_1} = \text{Class}_{\tau_2}$ .*

*Proof.* By Lemma 3.1.8 we have  $\theta_{\tau_1} = \theta_{\tau_2}$  if and only if  $\text{Class}_{\theta_{\tau_1}} = \text{Class}_{\theta_{\tau_2}}$ . But then as seen in the proof of Proposition 3.1.11 we have  $\text{Class}_{\tau_1} = \text{Class}_{\theta_{\tau_1}}$  and  $\text{Class}_{\tau_2} = \text{Class}_{\theta_{\tau_2}}$ . Thus  $\text{Class}_{\theta_{\tau_1}} = \text{Class}_{\theta_{\tau_2}}$  if and only if  $\text{Class}_{\tau_1} = \text{Class}_{\tau_2}$ .  $\square$

**Example 3.1.15.** In Example 3.1.7 we saw that for  $w = (3, 6, 9)$  and  $Q = P_{R_Q}$  with  $R_Q = R_{Q_w} = \{1, 2, 4, 5, 7, 8\}$  we had

$$\text{Head}_L = \{(1, 2, 3), (2, 3, 6), (2, 3, 9), (3, 5, 6), (3, 6, 9)\}.$$

If we draw the Hasse diagram for  $w$  and remove the edges labeled by  $s_{\alpha_m}$  for  $m \in R'_Q = \{3, 6\}$  we have the following disjoint diagram. The diagram has  $|\text{Head}_L| = 5$  disjoint subdiagrams, with the unique maximal element in each subdiagram a head of type  $L$ .



**Lemma 3.1.16.** *Given a  $\xi \in \text{Head}_L$ , there is a unique minimal element  $\tau$  such that  $\theta_\tau = \xi$ .*

*Proof.* Suppose  $\text{Class}_\xi = (s_1, \dots, s_d)$ . Define  $\tau = (j_1, \dots, j_d) \in H_w$  where  $j_m \in \text{Block}_{L, s_m}$  for  $1 \leq m \leq d$  and for all  $1 \leq k \leq d'_L$  we have that  $\tau \cap \text{Block}_{L, k}$  is either empty or contains a minimal collection

of elements from  $\text{Block}_{L,k}$ , explicitly for all  $m \in \tau \cap \text{Block}_{L,k}$  and all  $n \in \text{Block}_{L,k} \setminus \{\tau \cap \text{Block}_{L,k}\}$  we have  $m < n$ . Such a  $\tau$  is well defined and  $\text{Class}_\tau = \text{Class}_\xi$  implies, by Corollary 3.1.14, that  $\theta_\tau = \theta_\xi = \xi$ .

Further, if  $\phi \in H_w$  such that  $\theta_\phi = \xi$ , then Corollary 3.1.14 implies  $\text{Class}_\phi = \text{Class}_\tau$ . This means that  $|\tau \cap \text{Block}_{L,k}| = |\phi \cap \text{Block}_{L,k}|$  for  $1 \leq k \leq d'_L$ . But by its definition,  $\tau$  is minimal in each block, and so  $\tau \cap \text{Block}_{L,k} \leq \phi \cap \text{Block}_{L,k}$  for  $1 \leq k \leq d'_L$ . Thus  $\tau \leq \phi$ .  $\square$

**Remark 3.1.17.** In light of the above lemma, it is easy to see that  $\text{WStd}_\theta$  is in fact an interval in  $H_w$ .

Recall that a poset  $(A, \leq)$  is *self-dual* if  $(A, \leq) \cong (A, \leq^{op})$ , where the partial order  $\leq^{op}$  is defined for  $a, b \in A$  by  $a \leq^{op} b$  if and only if  $b \leq a$ . The *cartesian product* of the posets  $P$  and  $Q$  is the poset  $P \times Q$  on the set  $\{(p, q) \mid p \in P \text{ and } q \in Q\}$  along with the product order given by  $(p, q) \leq^{prod} (p', q')$  in  $P \times Q$  if  $p \leq p'$  in  $P$  and  $q \leq q'$  in  $Q$ .

**Proposition 3.1.18.** *Let  $\xi \in \text{Head}_L$  and  $\leq$  the Bruhat order.*

- (a) *The poset  $(\text{WStd}_\xi, \leq)$  is isomorphic to the cartesian product of posets associated to certain Grassmannians, along with the product order.*
- (b) *The poset  $(\text{WStd}_\xi, \leq)$  is a self-dual poset.*

*Proof.* (a) Define the partial order  $\leq^B$  on the set  $\text{WStd}_\xi$  to be  $\tau_1 \leq^B \tau_2$  if  $\text{Block}_{L,k} \cap \tau_1 \leq \text{Block}_{L,k} \cap \tau_2$  for all  $1 \leq k \leq d'_L$ , where  $\leq$  is the Bruhat order. Note that this is well-defined since  $|\text{Block}_{L,k} \cap \tau_1| = |\text{Block}_{L,k} \cap \tau_2|$ . This, in addition to the fact that the  $\text{Block}_{L,k}$  for  $1 \leq k \leq d'_L$  partition the set  $\{1, \dots, N\}$ , can be seen to imply that for  $\tau_1, \tau_2 \in \text{WStd}_\xi$  we have  $\tau_1 \leq \tau_2$  if and only if  $\tau_1 \leq^B \tau_2$ . Thus the poset  $(\text{WStd}_\xi, \leq) \cong (\text{WStd}_\xi, \leq^B)$ .

Let  $L_k := |\text{Block}_{L,k} \cap \xi|$ . Define

$$H_\xi^k = \begin{cases} \{()\} & \text{if } L_k = 0 \\ \{\gamma \in I_{L_k, N_k} \mid \gamma \leq (N_k - L_k + 1, \dots, N_k)\} & \text{if } L_k > 0 \end{cases}$$

where  $I_{L_k, N_k} := \{\underline{i} = (i_1, \dots, i_{L_k}) \mid 1 \leq i_1 < \dots < i_{L_k} \leq N_k\}$ . Note that when  $L_k \neq 0$ , we have  $H_\xi^k$  along with the Bruhat order is the poset associated to the Grassmannian  $X(N_k - L_k + 1, \dots, N) = G_{L_k, N_k}$ . Finally, for  $\tau \in \text{WStd}_\xi$  if  $|\text{Block}_{L,k} \cap \tau| = (i_m \leq \dots \leq i_n)$ , then define

$$\tau_k = \begin{cases} () & \text{if } L_k = 0 \\ (i_m - (N_1 + \dots + N_{k-1}), \dots, i_n - (N_1 + \dots + N_{k-1})) & \text{if } L_k > 0 \end{cases}.$$

Now consider the following map

$$\begin{aligned} g : \text{WStd}_\xi &\longrightarrow H_\xi^1 \times \dots \times H_\xi^{d'_L} \\ \tau &\longmapsto (\tau_1, \dots, \tau_{d'_L}) \end{aligned}$$

This map is well-defined since, for  $\tau_1, \tau_2 \in \text{WStd}_\xi$  and  $1 \leq k \leq d'_L$ , we have

$$|\text{Block}_{L,k} \cap \tau_1| = |\text{Block}_{L,k} \cap \tau_2|.$$

It is a quick check, using Lemma 3.1.16, to see that  $g$  is in fact a bijection. Further, under  $g$  we have  $(\text{WStd}_\xi, \leq^B) \cong (H_\xi^1 \times \dots \times H_\xi^{d'_L}, \leq^{prod})$ , where  $\leq^{prod}$  is the product order on the cartesian product of the posets  $(H_\xi^k, \leq)$ ,  $1 \leq k \leq d'_L$ .

(b) As remarked in part (a),  $(H_\xi^k, \leq)$  is either a single element set or the poset associated to the Grassmannian  $X(N_k - L_k + 1, \dots, N) = G_{L_k, N_k}$ . In either case,  $(H_\xi^k, \leq)$  is self-dual. If  $(P, \leq^P), (Q, \leq^Q)$  are both self-dual posets, then the set  $P \times Q$  along with the product order is self-dual, and this

extends inductively to an arbitrary finite cartesian product of self-dual posets with the product order. Thus  $(H_\xi^1 \times \cdots \times H_\xi^{d_L}, \leq^{\text{prod}}) \cong (\text{WStd}_\xi, \leq^B) \cong (\text{WStd}_\xi, \leq)$  is self-dual.  $\square$

**Lemma 3.1.19.** *Let  $\theta_1 := (p_1, \dots, p_d), \theta_2 := (q_1, \dots, q_d) \in \text{Head}_L$ . Let  $\text{Class}_{\theta_1} := (s_1, \dots, s_d)$  and  $\text{Class}_{\theta_2} := (t_1, \dots, t_d)$ . If  $s_j \geq t_j$  for all  $1 \leq j \leq d$ , then  $\theta_1 \geq \theta_2$ .*

*Proof.* Suppose that  $\theta_1 \not\geq \theta_2$ . This implies there must be an index  $k$  such that  $q_k > p_k$ . We have by our hypothesis that  $s_k \geq t_k$ . If  $s_k > t_k$  then  $p_k > q_k$ , which is not the case by our assumption so  $s_k = t_k$ .

Let  $m$  be the maximum integer such that  $s_k = s_m$ , and  $n$  the maximum integer such that  $t_k = t_n$ . We know  $n \geq m$ , otherwise  $m > n$  would imply  $t_m > s_m$ , which is a contradiction of our hypothesis.

But then  $p_k, \dots, p_m$  and  $q_k, \dots, q_n$  are both maximal sequences in  $\text{Block}_{L, s_k}$  by Proposition 3.1.5(iii). However, the length of the sequence  $q_k, \dots, q_n$  is longer or equal to  $p_k, \dots, p_m$ , so  $p_k \geq q_k$ . This is a contradiction as  $k$  was chosen to be the index where  $q_k > p_k$ . Thus it must be the case that  $\theta_1 \geq \theta_2$ .  $\square$

**3.2. A Partial Order on the Set of Degree  $r$  Heads.** We may extend the definition of a head in the following way.

**Definition 3.2.1.** Let  $\tau_1, \dots, \tau_r \in H_w$ . Then define the degree  $r$  head of  $(\tau_1, \dots, \tau_r)$  to be  $(\theta_{\tau_1}, \dots, \theta_{\tau_r})$ . This degree  $r$  head is clearly unique since each individual head is unique.

To each  $\theta \in \text{Head}_L$  we may associate a collection of Plücker coordinates  $p_\tau$  such that  $\tau$  has head  $\theta$ . This gives us a partition of the degree 1 standard monomials by Proposition 3.1.11. The next step is to describe a partition of the degree  $r$  standard monomials in terms of degree  $r$  heads (cf. Corollary 3.2.6). The fact that this is possible is due to a remarkable property of the degree 1 heads: given two elements  $\tau_1, \tau_2 \in H_w$  which satisfy  $\tau_1 \geq \tau_2$ , their respective degree 1 heads  $\theta_{\tau_1}, \theta_{\tau_2}$  satisfy  $\theta_{\tau_1} \geq \theta_{\tau_2}$  (as we shall see in Proposition 3.2.2 for the case  $r = 1$ ). Note that this property does not hold for any partition of the Hasse diagram, or even any partition with each subdiagram containing a unique maximal element.

**Proposition 3.2.2.** *Let  $\tau_1, \dots, \tau_r \in H_w$ ,  $\tau_1 \geq \dots \geq \tau_r$ , with degree  $r$  head  $\theta_{\tau_1}, \dots, \theta_{\tau_r}$ . Then  $\theta_{\tau_1} \geq \dots \geq \theta_{\tau_r}$ .*

*Proof.* Let  $i$  be an integer with  $1 \leq i < r$ . We have that  $\tau_i := (x_1, \dots, x_d) \geq \tau_{i+1} := (y_1, \dots, y_d)$ . Let  $\text{Class}_{\tau_i} := (s_1, \dots, s_d)$  and  $\text{Class}_{\tau_{i+1}} := (t_1, \dots, t_d)$ . Suppose that  $t_j > s_j$  for some  $1 \leq j \leq d$ , then this would imply that  $y_j > x_j$  which is a contradiction of  $\tau_i \geq \tau_{i+1}$ . Thus

$$(3.2.3) \quad s_j \geq t_j \text{ for all } 1 \leq j \leq d.$$

By Proposition 3.1.11  $\text{Class}_{\theta_{\tau_i}} = \text{Class}_{\tau_i} = (s_1, \dots, s_d)$  and  $\text{Class}_{\theta_{\tau_{i+1}}} = \text{Class}_{\tau_{i+1}} = (t_1, \dots, t_d)$ .

This together with (3.2.3) implies, by Lemma 3.1.19, that  $\theta_{\tau_i} \geq \theta_{\tau_{i+1}}$ . As our choice of  $i$  was arbitrary we are done.  $\square$

**Definition 3.2.4.** Define

$$\text{WStd}_r := \{(\tau_1, \dots, \tau_r) \mid \tau_i \in H_w \text{ for } 1 \leq i \leq r \text{ and } \tau_1 \geq \dots \geq \tau_r\}$$

and

$$\text{Std}_r := \{p_{\tau_1} \cdots p_{\tau_r} \mid (\tau_1, \dots, \tau_r) \in \text{WStd}_r\}.$$

In a slight abuse of terminology, we will call the sequences in  $\text{WStd}_r$  standard. Define the set of all *standard degree  $r$  heads* to be

$$\text{Head}_{L,r} := \{(\theta_1, \dots, \theta_r) \in \text{WStd}_r \mid \theta_i \in \text{Head}_L\}.$$

And finally for  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$  define

$$\text{WStd}_{\underline{\theta}} := \{(\tau_1, \dots, \tau_r) \in \text{WStd}_r \mid \tau_i \text{ has head } \theta_i \text{ for } 1 \leq i \leq r\}$$

and

$$\text{Std}_{\underline{\theta}} := \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid (\tau_1, \dots, \tau_r) \in \text{WStd}_{\underline{\theta}}\}$$

Note that the above definition agrees with Definition 3.1.13 when  $r = 1$ . Further note that  $\text{Head}_{L,1} = \text{Head}_L$ .

We will often want to refer to the subspace of  $\mathbb{C}[X(w)]_r$  generated by these sets; for  $X \subseteq \text{Std}_r$  let  $\langle X \rangle$  denote the span of the elements in  $X$ .

With these definitions in hand we may now state two important corollaries of Proposition 3.2.2.

**Corollary 3.2.5.** *Let  $p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r$ . Then  $p_{\theta_{\tau_1}} \cdots p_{\theta_{\tau_r}}$  is standard.*

**Corollary 3.2.6.** *The set  $\text{Std}_r$  is partitioned into disjoint subsets labeled by  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$ . Explicitly*

$$\text{Std}_r = \bigsqcup_{\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}} \text{Std}_{\underline{\theta}}.$$

And this implies

$$\langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}} \langle \text{Std}_{\underline{\theta}} \rangle.$$

*Proof.* This is immediate by Proposition 3.2.2 since each degree  $r$  standard monomial  $p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r$  has a unique standard monomial  $p_{\theta_{\tau_1}} \cdots p_{\theta_{\tau_r}}$  such that  $(\theta_{\tau_1}, \dots, \theta_{\tau_r}) \in \text{Head}_{L,r}$  is the degree  $r$  head of  $(\tau_1, \dots, \tau_r)$ .  $\square$

When the degree is equal to 1 the  $\langle \text{Std}_{\underline{\theta}} \rangle$  with  $\underline{\theta} \in \text{Head}_{L,1} (= \text{Head}_L)$  are  $L$ -stable, and in fact are irreducible  $L$ -modules (cf. Remark 3.5.8). Our initial hope was that this might extend to higher degrees. Unfortunately, when  $r > 1$ , it is no longer the case that the  $\langle \text{Std}_{\underline{\theta}} \rangle$  are  $L$ -stable for all  $\underline{\theta} \in \text{Head}_{L,r}$ . This is due to the interaction between the  $L$ -action and the standard monomial straightening process. To correct for this lack of  $L$ -stability we introduce a partial order on the set of degree  $r$  heads, inspired by the straightening process, which will allow us to introduce new subspaces of  $\langle \text{Std}_r \rangle$  that are  $L$ -stable.

**Definition 3.2.7.** We now define a partial order on the set of (standard) degree  $r$  heads  $\text{Head}_{L,r}$  that, as we will see in Theorem 3.3.4, is closely related to the straightening rule and hence shall denote it  $\geq_{str}$ . Define  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta'_1, \dots, \theta'_r)$  if there exists a  $q \leq r$  such that  $\theta_q > \theta'_q$  and  $\theta_l = \theta'_l$  for all  $l > q$ . Equality occurs if  $\theta_l = \theta'_l$  for all  $1 \leq l \leq r$ .

This satisfies the following properties.

(reflexivity)  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta_1, \dots, \theta_r)$

(antisymmetry) Suppose that  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta'_1, \dots, \theta'_r)$  and  $(\theta'_1, \dots, \theta'_r) \geq_{str} (\theta_1, \dots, \theta_r)$ . Suppose that  $(\theta_1, \dots, \theta_r) \neq_{str} (\theta'_1, \dots, \theta'_r)$ . Then there exists a  $q_1$  such that  $\theta_{q_1} > \theta'_{q_1}$  and  $\theta_l = \theta'_l$  for all  $l > q_1$ . Also there exists a  $q_2$  such that  $\theta'_{q_2} > \theta_{q_2}$  and  $\theta'_l = \theta_l$  for all  $l > q_2$ . This is clearly not possible and thus  $(\theta_1, \dots, \theta_r) =_{str} (\theta'_1, \dots, \theta'_r)$ .

(transitivity) Suppose that  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta'_1, \dots, \theta'_r)$  and  $(\theta'_1, \dots, \theta'_r) \geq_{str} (\theta''_1, \dots, \theta''_r)$ . This is trivially transitive if either of these is equality; so assume neither is. Then there exists a  $q_1$  such that  $\theta_{q_1} > \theta'_{q_1}$  and  $\theta_l = \theta'_l$  for all  $l > q_1$ . Also there exists a  $q_2$  such that  $\theta'_{q_2} > \theta''_{q_2}$  and  $\theta'_l = \theta''_l$  for all  $l > q_2$ . If  $q_1 > q_2$  then  $\theta_{q_1} > \theta'_{q_1} = \theta''_{q_1}$  and  $\theta_l = \theta'_l = \theta''_l$  for all  $l > q_1$ . Thus  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta''_1, \dots, \theta''_r)$ . If  $q_2 \geq q_1$  then  $\theta_{q_2} \geq \theta'_{q_2} > \theta''_{q_2}$  and  $\theta_l = \theta'_l = \theta''_l$  for all  $l > q_2$ . Thus  $(\theta_1, \dots, \theta_r) \geq_{str} (\theta''_1, \dots, \theta''_r)$ .

Thus  $\geq_{str}$  is a partial order. In fact  $\geq_{str}$  is the reverse lexicographic order (corresponding to the Bruhat order  $\geq$ ).

**Definition 3.2.8.** Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$ . Define

$$\text{Std}_{\underline{\theta}}^{\geq str} = \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid \underline{\theta} \geq_{str} (\theta_{\tau_1}, \dots, \theta_{\tau_r})\}$$

and

$$\text{Std}_{\underline{\theta}}^{> str} = \{p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r \mid \underline{\theta} >_{str} (\theta_{\tau_1}, \dots, \theta_{\tau_r})\}.$$

**Remark 3.2.9.** Note that

$$\text{Std}_{\underline{\theta}} = \text{Std}_{\underline{\theta}}^{\geq str} \setminus \text{Std}_{\underline{\theta}}^{> str}$$

which implies that

$$\langle \text{Std}_{\underline{\theta}} \rangle \cong \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle.$$

The goal of the next section is to show that  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  and  $\langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  are both  $L$ -stable.

**3.3. The Relation between  $\geq_{str}$  and the  $\mathfrak{l}$ -action.** Let  $E_{ij}$  be the  $N \times N$  matrix with a 1 in the  $(i, j)$ th entry, and zero in all other entries. The action of  $\mathfrak{l} \subset \mathfrak{gl}_N$  on  $\mathbb{C}[X(w)]$  is induced by the action of  $\mathfrak{gl}_N$  on  $\mathbb{C}[X(w)]$ . The generators of  $\mathfrak{l}$  as a Lie algebra are

$$X_i = E_{ii+1}, \quad X_{-i} = E_{i+1i}, \quad H_i = E_{ii} \text{ for } i \in R_Q$$

satisfying the relations

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, X_{\pm j}] &= \pm X_{\pm j} \\ [X_i, X_{-j}] &= \delta_{ij}(H_i - H_{i+1}) \\ (ad X_{\pm i})^{1-a_{ij}} X_{\pm i} &= 0 \end{aligned}$$

where  $\delta_{ij}$  is the Kronecker delta and  $a_{ij} = 2\delta_{ij} - \delta_{i-1j} - \delta_{i+1j}$ . Let  $\tau = (i_1, \dots, i_d) \in H_w$  and denote the integers  $\{1, \dots, N\} \setminus \{i_1, \dots, i_d\}$  by  $j_1, \dots, j_{N-d}$  (arranged in ascending order). We then identify the Plücker coordinate  $p_{\tau} \in \mathbb{C}[X(w)]$  with the element  $e_{j_1} \wedge \dots \wedge e_{j_{N-d}} \in \bigwedge^{N-d}(\mathbb{C}^N)$ . Using this identification and the fact that

$$E_{ij}e_k = \delta_{jk}e_i$$

we may calculate the action of the algebra generators of  $\mathfrak{l}$  on a Plücker coordinate.

Let  $p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_r$ . Then for  $i \in R_Q$  the action on a degree  $r$  standard monomial is given by

$$\begin{aligned} X_{\pm i}(p_{\tau_1} \cdots p_{\tau_r}) &= \sum_{j=1}^r p_{\tau_1} \cdots X_{\pm i}(p_{\tau_j}) \cdots p_{\tau_r} \\ H_i(p_{\tau_1} \cdots p_{\tau_r}) &= \sum_{j=1}^r p_{\tau_1} \cdots H_i(p_{\tau_j}) \cdots p_{\tau_r} \end{aligned}$$

Then

$$X_i(p_{\tau_j}) = \begin{cases} p_{s_{\alpha_i}\tau_j} & \text{if } \tau_j \text{ has an entry equal to } i \text{ and no entry equal to } i+1 \\ 0 & \text{otherwise} \end{cases}$$

where in the single nonzero case  $s_{\alpha_i}\tau_j$  is obtained from  $\tau_j$  by replacing  $i$  with  $i+1$ . And

$$X_{-i}(p_{\tau_j}) = \begin{cases} p_{s_{\alpha_i}\tau_j} & \text{if } \tau_j \text{ has an entry equal to } i+1 \text{ and no entry equal to } i \\ 0 & \text{otherwise} \end{cases}$$



where in the single nonzero case  $s_{\alpha_i}\tau_j$  is obtained from  $\tau_j$  by replacing  $i + 1$  with  $i$ . Finally

$$H_i(p_{\tau_j}) = \begin{cases} p_{\tau_j} & \text{if } \tau_j \text{ has no entry equal to } i \\ 0 & \text{otherwise} \end{cases}$$

**Remark 3.3.1.** We are primarily interested in these results for checking the  $L$ -stability of subspaces of  $\mathbb{C}[X(w)]$ , and since such a subspace is  $L$ -stable if and only if it is  $\mathfrak{l}$ -stable for the induced action, we may reduce to checking stability under the Lie algebra action. The benefit of this is that the Lie algebra action is easier to calculate. Note also that the action of the  $H_i$  on a Plücker coordinate always is either zero or the Plücker coordinate itself, and thus a subspace that has a basis of Plücker coordinates will always be stable under the action of the  $H_i$ .

We would like to investigate the interplay between the above action, the straightening algorithm, and the partial order described in Section 3.2.

**Remark 3.3.2.** Let  $i \in R_Q$ ,  $\tau \in H_w$ . Suppose  $X_{\pm\alpha_i}(p_\tau)$  is nonzero and let  $\tau' = s_{\alpha_i}\tau_j$ . Then  $p_{\tau'} = X_{\pm\alpha_i}(p_\tau)$ . In the case of  $X_{\alpha_i}$  we saw that  $s_{\alpha_i}\tau$  is obtained from  $\tau$  by replacing  $i$  with  $i + 1$ , and in the case of  $X_{-\alpha_i}$  we saw that  $s_{\alpha_i}\tau$  is obtained from  $\tau$  by replacing  $i + 1$  with  $i$ . In both cases  $i$  and  $i + 1$  are in the same block since  $i \in R_Q$  which implies that  $\text{Class}_\tau = \text{Class}_{\tau'}$ . Thus  $\tau$  and  $\tau'$  have the same head.

**Lemma 3.3.3.** Let  $\tau = (\tau_1, \dots, \tau_d), \phi = (\phi_1, \dots, \phi_d) \in H_w$  with  $\tau \not\leq \phi$  in the Bruhat order. Let

$$p_\tau p_\phi = \sum_{\alpha, \beta \in \text{WStd}_2} A_{\alpha, \beta} p_\alpha p_\beta \text{ with } A_{\alpha, \beta} \in \mathbb{C}$$

be the expression for  $p_\tau p_\phi$  as a sum of standard monomials on  $X(w)$ . We have:

- (a) For every  $\alpha, \beta$  such that  $A_{\alpha, \beta} \neq 0$  we have that  $\alpha > \tau$  and  $\beta < \phi$ .
- (b) The heads  $\theta_\beta = \theta_\phi$  if and only if  $\theta_\alpha = \theta_\tau$ . Otherwise  $\theta_\alpha > \theta_\tau$  and  $\theta_\beta < \theta_\phi$ .

*Proof.* (a) The fact that  $\alpha > \tau$  and  $\beta < \phi$  is implied by the details of the straightening process discussed in Section 2.2.

(b) Suppose that

$$p_\tau p_\phi = \sum_{\alpha, \beta} A_{\alpha, \beta} p_\alpha p_\beta \text{ with } A_{\alpha, \beta} \in \mathbb{C},$$

is the expression for  $p_\tau p_\phi$  as a sum of standard monomials on  $X(w)$ . Following Section 2.2 and (2.2.1) we get that for each  $\alpha, \beta$  with  $A_{\alpha, \beta} \neq 0$  we have  $\alpha = (((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}$ ,  $\beta = (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}$  for some  $M > 0$  and some  $\sigma_1 \in [\tau, \phi], \dots, \sigma_M \in [(((\alpha^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}, (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}]$ .

Fix an arbitrary  $\alpha, \beta$  with  $A_{\alpha, \beta} \neq 0$ . Then as we noted in Section 2.2 we have

$$\beta = (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M} < (((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}} < \dots < \beta^{\sigma_1} < \phi.$$

By Proposition 3.2.2 this implies that

$$\theta_\beta = \theta_{(((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}} \leq \theta_{(((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}} \leq \dots \leq \theta_{\beta^{\sigma_1}} \leq \theta_\phi.$$

Now suppose  $\theta_\beta = \theta_\phi$ . Then

$$\theta_\beta = \theta_{(((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_M}} = \theta_{(((\beta^{\sigma_1})^{\sigma_2}) \dots)^{\sigma_{M-1}}} = \dots = \theta_{\beta^{\sigma_1}} = \theta_\phi.$$

We will first show that this implies  $\theta_{\alpha\sigma_1} = \theta_\tau$ . Since  $\theta_{\beta\sigma_1} = \theta_\phi$  we have by Corollary 3.1.14 that  $\text{Class}_{\beta\sigma_1} = \text{Class}_\phi$ , that is the class of  $(\phi_1, \dots, \phi_d)$  equals the class of  $(\phi_1, \dots, \phi_{t-1}, \sigma_1(\phi_t), \dots, \sigma_1(\phi_d)) \uparrow$ . This implies the class of  $(\phi_t, \dots, \phi_d)$  equals the class of  $(\sigma_1(\phi_t), \dots, \sigma_1(\phi_d))$ . Thus these two sequences have the same number of entries in  $\text{Block}_{L,k}$  for  $1 \leq k \leq d'_L$ .

Suppose now that the class of  $(\sigma_1(\tau_1), \dots, \sigma_1(\tau_t)) \uparrow$  does not equal the class of  $(\tau_1, \dots, \tau_t)$ . This implies that these two sequences have a different number of entries in  $\text{Block}_{L,j}$  for some  $1 \leq j \leq d'_L$ . But this, combined with the fact that  $(\phi_t, \dots, \phi_d)$  and  $(\sigma_1(\phi_t), \dots, \sigma_1(\phi_d))$  have the same number of entries in  $\text{Block}_{L,j}$ , means that  $\{\sigma_1(\tau_1), \dots, \sigma_1(\tau_t), \sigma_1(\phi_t), \dots, \sigma_1(\phi_d)\}$  and  $\{\tau_1, \dots, \tau_t, \phi_t, \dots, \phi_d\}$  have a different number of entries in  $\text{Block}_{L,j}$ . This is a contradiction of the fact that the multiset  $\{\sigma_1(\tau_1), \dots, \sigma_1(\tau_t), \sigma_1(\phi_t), \dots, \sigma_1(\phi_d)\}$  is a permutation of the multiset  $\{\tau_1, \dots, \tau_t, \phi_t, \dots, \phi_d\}$ .

Thus the class of  $(\tau_1, \dots, \tau_t)$  equals the class of  $(\sigma_1(\tau_1), \dots, \sigma_1(\tau_t))$ . This implies the class of  $\alpha^{\sigma_1} = (\sigma_1(\tau_1), \dots, \sigma_1(\tau_t), \tau_{t+1}, \dots, \tau_d) \uparrow$  equals the class of  $(\tau_1, \dots, \tau_d)$ . But by Corollary 3.1.14 this implies  $\theta_{\alpha\sigma_1} = \theta_\tau$ . The fact that  $\theta_\alpha = \theta_{((\alpha\sigma_1)\sigma_2)\dots\sigma_M} = \theta_\tau$  follows by extending this argument inductively in the obvious way.

The converse follows by an analogous argument.

To finish the proof of this lemma we note that by Proposition 3.2.2  $\beta < \phi$  implies that  $\theta_\beta \leq \theta_\phi$  and  $\alpha > \tau$  implies that  $\theta_\alpha \geq \theta_\tau$ . So if  $\theta_\beta \neq \theta_\phi$  and  $\theta_\alpha \neq \theta_\tau$  we must have  $\theta_\beta < \theta_\phi$  and  $\theta_\alpha > \theta_\tau$ .  $\square$

**Theorem 3.3.4.** *Let  $(\tau_1, \dots, \tau_r) \in \text{WStd}_r$  with degree  $r$  head  $\underline{\theta} = (\theta_{\tau_1}, \dots, \theta_{\tau_r})$ . Let  $i \in R_Q$ . If*

$$X_{\pm\alpha_i}(p_{\tau_1} \cdots p_{\tau_r}) = \sum_{\gamma_1, \dots, \gamma_r \in \text{WStd}_r} A_{\gamma_1, \dots, \gamma_r} p_{\gamma_1} \cdots p_{\gamma_r} \text{ with } A_{\gamma_1, \dots, \gamma_r} \in \mathbb{C}$$

*is the expression for  $X_{\pm\alpha_i}(p_{\tau_1} \cdots p_{\tau_r})$  as a sum of standard monomials, then  $(\theta_{\tau_1}, \dots, \theta_{\tau_r}) \geq_{str} (\theta_{\gamma_1}, \dots, \theta_{\gamma_r})$  for all  $\gamma_1, \dots, \gamma_r \in \text{WStd}_r$  such that  $A_{\gamma_1, \dots, \gamma_r} \neq 0$ .*

*Proof.* By definition

$$X_{\pm\alpha_i}(p_{\tau_1} \cdots p_{\tau_r}) = \sum_{j=1}^r p_{\tau_1} \cdots X_{\pm\alpha_i}(p_{\tau_j}) \cdots p_{\tau_r}.$$

Any  $(\gamma_1, \dots, \gamma_r) \in \text{WStd}_r$  such that  $A_{\gamma_1, \dots, \gamma_r} \neq 0$  is the result of repeated applications of the degree 2 straightening process on one of the above summands. Thus our goal is to show that any standard monomial that is the result of the degree 2 straightening process on one of the above summands will have the property that its associated degree  $r$  head is less than  $\underline{\theta}$  in the partial order  $\geq_{str}$ .

Fix a  $j$  where  $X_{\pm\alpha_i}(p_{\tau_j}) \neq 0$  and set  $\tau_l^1 = \tau_l$  for  $l \neq j$  and  $\tau_j^1 = s_{\alpha_i}\tau_j$ . Then  $(\tau_1^1, \dots, \tau_r^1) = (\tau_1, \dots, \tau_{j-1}, s_{\alpha_i}\tau_j, \tau_{j+1}, \dots, \tau_r)$ . Since  $i \in R_Q$  we have that the degree  $r$  head of  $(\tau_1^1, \dots, \tau_r^1)$  is  $(\theta_{\tau_1}, \dots, \theta_{\tau_r})$  by Remark 3.3.2.

Thus if  $p_{\tau_1^1} \cdots p_{\tau_r^1}$  is standard we are done. Otherwise, if  $p_{\tau_1^1} \cdots p_{\tau_r^1}$  is not standard then there is a  $k$ , such that  $\tau_k^1 \not\leq \tau_{k+1}^1$ . Then by Lemma 3.3.3(a) we have that if

$$p_{\tau_k^1} p_{\tau_{k+1}^1} = \sum_{\alpha, \beta \in \text{WStd}_2} A_{\alpha, \beta} p_\alpha p_\beta \text{ with } A_{\alpha, \beta} \in \mathbb{C}$$

is the expression for  $p_{\tau_k^1} p_{\tau_{k+1}^1}$  as a sum of standard monomials then for every  $\alpha, \beta$  such that  $A_{\alpha, \beta} \neq 0$  we have that  $\alpha > \tau_k^1$  and  $\beta < \tau_{k+1}^1$ .

Substituting this sum of standard monomials for  $p_{\tau_k^1} p_{\tau_{k+1}^1}$  in  $p_{\tau_1^1} \cdots p_{\tau_r^1}$  we have

$$p_{\tau_1^1} \cdots p_{\tau_r^1} = \sum_{\alpha, \beta \in \text{WStd}_2} A_{\alpha, \beta} p_{\tau_1^1} \cdots p_{\tau_{k-1}^1} p_\alpha p_\beta p_{\tau_{k+2}^1} \cdots p_{\tau_r^1}.$$

We now choose an arbitrary summand from this summation and continue the straightening process. Choose an  $\alpha, \beta \in \text{WStd}_2$  such that  $A_{\alpha, \beta} \neq 0$  and set  $\tau_l^2 = \tau_l^1$  for  $l \neq k, k+1$  and  $\tau_k^2 = \alpha$  and  $\tau_{k+1}^2 = \beta$ . Then  $(\tau_1^2, \dots, \tau_r^2) = (\tau_1^1, \dots, \tau_{k-1}^1, \alpha, \beta, \tau_{k+2}^1, \dots, \tau_r^1)$ . Note that  $\tau_{k+1}^2 = \beta < \tau_{k+1}^1$  and  $\tau_k^2 = \alpha > \tau_k^1$ .

To keep track of the degree  $r$  head as we continue this process we create a label in the following way. Start with a blank label  $(, \dots, )$  with  $r$  empty entries. Then we have two cases according to Lemma 3.3.3(b).

Case 1:  $\theta_{\tau_k^2} = \theta_{\tau_k^1}$  and  $\theta_{\tau_{k+1}^2} = \theta_{\tau_{k+1}^1}$ . In this case we append a "-" to the  $k$ -th and  $k+1$ -th entry of the label, and leave the other entries unchanged.

Case 2:  $\theta_{\tau_k^2} > \theta_{\tau_k^1}$  and  $\theta_{\tau_{k+1}^2} < \theta_{\tau_{k+1}^1}$ . In this case we append a " $\wedge$ " to the  $k$ -th entry of the label and append a " $\vee$ " to the  $k+1$ -th entry of the label, and leave the other entries unchanged.

Now we repeat this same process inductively, replacing a nonstandard degree 2 piece with an arbitrary summand from its straightening, continuing to append the "-", " $\wedge$ ", and " $\vee$ " to the labeling to keep track of the degree  $r$  heads.

Eventually after say  $M$  steps  $p_{\tau_1^M} \cdots p_{\tau_r^M}$  will be standard. There will be  $2M$  "-", " $\wedge$ ", and " $\vee$ " distributed among the entries of the label. However due to the way we have replaced there will always be a rightmost entry  $m$  in the label such that the  $m$ th entry contains only "-" and " $\vee$ ", and every entry to the right contains only "-". This must be true since the  $r$ th entry can only ever contain "-" and " $\vee$ ".

If the label contains only "-", that is no " $\vee$ " in any entry, then we know that  $\theta_{\tau_l^M} = \theta_{\tau_l}$  for all  $1 \leq l \leq r$  and thus we have that the degree  $r$  head of  $(\tau_1^M, \dots, \tau_r^M)$  equals  $(\theta_{\tau_1}, \dots, \theta_{\tau_r})$ .

Otherwise if there is at least one " $\vee$ " in any entry then we know that  $\theta_{\tau_m^M} < \theta_{\tau_m}$  and  $\theta_{\tau_l^M} = \theta_{\tau_l}$  for all  $m+1 \leq l \leq r$ . But this precisely means that  $(\theta_{\tau_1}, \dots, \theta_{\tau_r}) \geq_{str} (\theta_{\tau_1^M}, \dots, \theta_{\tau_r^M})$ .

And thus we have shown that for any standard monomial  $p_{\tau_1^M} \cdots p_{\tau_r^M}$  resulting from the straightening process, we will have  $(\theta_{\tau_1}, \dots, \theta_{\tau_r}) \geq_{str} (\theta_{\tau_1^M}, \dots, \theta_{\tau_r^M})$ . □

**Corollary 3.3.5.** *Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$ . Then  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  and  $\langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  are  $L$ -stable.*

*Proof.* This follows by Theorem 3.3.4 and Remark 3.3.1. □

**3.4. The Skew Semistandard Tableaux associated to a degree  $r$  standard monomial.** Before we can give our decomposition of  $\mathbb{C}[X(w)]$  we need to describe a method for associating a degree  $r$  standard monomial to a collection of semi-standard young tableaux.

Let  $\underline{\tau} = (\tau_1, \dots, \tau_r) \in \text{WStd}_r$ . Define the semistandard tableaux  $T_{\underline{\tau}}$  on the diagram  $(r^d)$  by letting the columns of  $T_{\underline{\tau}}$  correspond to the  $\tau_i$  in  $\underline{\tau}$ , but with their order reversed. Thus the standardness of  $\underline{\tau}$  implies that this tableaux is semistandard.

**Example 3.4.1.** Let  $d = 3$  and  $N = 9$ . Consider  $w = (3, 6, 9) \in W^{P_3}$ . Suppose  $\underline{\tau} := (\tau_1, \tau_2, \tau_3) = ((3, 5, 9), (2, 3, 8), (1, 2, 4))$ . Then

$$T_{\underline{\tau}} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 5 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$

We would now like to associate a skew semistandard tableaux  $T_{\underline{\tau}}^{(k)}$  to  $\underline{\tau}$  for each  $\text{Block}_{L,k}$ ,  $1 \leq k \leq d'_L$ . To do this we start by fixing a  $k$ ,  $1 \leq k \leq d'_L$ , and then define  $\widehat{T}_{\underline{\tau}}^{(k)}$  to be the skew semistandard tableaux created by deleting all boxes with values not in  $\text{Block}_{L,k}$ . It is not immediately apparent that such an operation will result in  $\widehat{T}_{\underline{\tau}}^{(k)}$  having a shape that is a skew

diagram. However the only way that the shape could fail to be a skew diagram is if one of two possibilities occur:

- (i) For some  $i < j$ , the maximum column index containing a value in  $\text{Block}_{L,k}$  in row  $i$  is less than the maximum column index containing a value in  $\text{Block}_{L,k}$  in row  $j$ .
- (ii) For some  $i < j$ , the minimum column index containing a value in  $\text{Block}_{L,k}$  in row  $i$  is less than the minimum column index containing a value in  $\text{Block}_{L,k}$  in row  $j$ .

But we shall now see that if either of these occur it would imply that  $T_{\underline{\tau}}$  is not semistandard. Suppose that possibility (i) occurs. Explicitly, set  $i < j$  and let  $m_i$  and  $m_j$  be the maximum column indexes containing values in  $\text{Block}_{L,k}$  for rows  $i$  and  $j$  respectively. Then suppose  $m_i < m_j$ . We know that  $T_{\underline{\tau}}(i, m_j)$  is not in  $\text{Block}_{L,k}$  and the fact that  $T_{\underline{\tau}}$  is semistandard implies that  $T_{\underline{\tau}}(i, m_j) < T_{\underline{\tau}}(j, m_j)$ . Thus  $T_{\underline{\tau}}(i, m_j)$  is in  $\text{Block}_{L,k'}$  for some  $k' < k$ . But  $T_{\underline{\tau}}$  semistandard also implies that  $T_{\underline{\tau}}(i, m_i) \leq T_{\underline{\tau}}(i, m_j)$ , which implies that  $T_{\underline{\tau}}(i, m_j)$  is in  $\text{Block}_{L,k''}$  for some  $k'' > k$ . This is a contradiction. An analogous argument generates a contradiction in the case of possibility (ii).

Thus the shape of  $\hat{T}_{\underline{\tau}}^{(k)}$  is of the form  $\hat{\lambda}_{\underline{\tau}}^{(k)} / \hat{\mu}_{\underline{\tau}}^{(k)}$  for some partitions  $\hat{\mu}_{\underline{\tau}}^{(k)}, \hat{\lambda}_{\underline{\tau}}^{(k)}$  with  $\hat{\mu}_{\underline{\tau}}^{(k)} \subseteq \hat{\lambda}_{\underline{\tau}}^{(k)}$ . Then let  $\hat{\tilde{T}}_{\underline{\tau}}^{(k)}$  be the tableaux on the skew diagram  $\lambda_{\underline{\tau}}^{(k)} / \mu_{\underline{\tau}}^{(k)}$  that is defined by deleting the empty rows and columns from the tableaux  $\hat{T}_{\underline{\tau}}^{(k)}$  on  $\hat{\lambda}_{\underline{\tau}}^{(k)} / \hat{\mu}_{\underline{\tau}}^{(k)}$ .

Every entry in  $\hat{\tilde{T}}_{\underline{\tau}}^{(k)}$  has a value in  $\text{Block}_{L,k}$ , that is they have values that range from  $\hat{a}_k + 1$  to  $\hat{a}_{k+1}$  (cf. Section 3.1). As our final step in the process we set  $T_{\underline{\tau}}^{(k)}$  equal to  $\hat{\tilde{T}}_{\underline{\tau}}^{(k)}$  with  $\hat{a}_k$  subtracted from every box.

Thus  $T_{\underline{\tau}}^{(k)}$  is a tableaux of shape  $\lambda_{\underline{\tau}}^{(k)} / \mu_{\underline{\tau}}^{(k)}$  with each entry taking on values from 1 to  $\hat{a}_{k+1} - \hat{a}_k = N_k$ .

The total number of boxes in the skew partitions  $\lambda_{\underline{\tau}}^{(k)} / \mu_{\underline{\tau}}^{(k)}$  for  $1 \leq k \leq d'_L$  is equal to the number of boxes in the tableaux  $T_{\underline{\tau}}$ . Thus we have

$$(3.4.2) \quad (|\lambda_{\underline{\tau}}^{(1)}| - |\mu_{\underline{\tau}}^{(1)}|) + \cdots + (|\lambda_{\underline{\tau}}^{(d'_L)}| - |\mu_{\underline{\tau}}^{(d'_L)}|) = rd$$

**Example 3.4.3.** Let us construct the partitions and tableaux associated to the standard monomial  $p_{\tau_1} p_{\tau_2} p_{\tau_3}$  with  $\underline{\tau} := (\tau_1, \tau_2, \tau_3) = ((3, 5, 9), (2, 3, 8), (1, 2, 4))$  as in Example 3.4.1. In Example 3.1.7, we saw that choosing  $Q = Q_w$  gave us  $\text{Block}_{L,1} = (1, 2, 3)$ ,  $\text{Block}_{L,2} = (4, 5, 6)$ , and  $\text{Block}_{L,3} = (7, 8, 9)$ . Then

$$T_{\underline{\tau}} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 5 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$

and deleting boxes from different blocks gives

$$\hat{T}_{\underline{\tau}}^{(1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \hline \end{array} \quad \hat{T}_{\underline{\tau}}^{(2)} = \begin{array}{|c|} \hline 5 \\ \hline 4 \\ \hline \end{array} \quad \hat{T}_{\underline{\tau}}^{(3)} = \begin{array}{|c|c|} \hline 8 & 9 \\ \hline \end{array}$$

and deleting empty rows and columns gives

$$\hat{\tilde{T}}_{\underline{\tau}}^{(1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \hline \end{array} \quad \hat{\tilde{T}}_{\underline{\tau}}^{(2)} = \begin{array}{|c|} \hline 5 \\ \hline 4 \\ \hline \end{array} \quad \hat{\tilde{T}}_{\underline{\tau}}^{(3)} = \begin{array}{|c|c|} \hline 8 & 9 \\ \hline \end{array}$$

and finally subtracting  $\hat{a}_k$  from each entry gives the final skew semistandard Young tableaux

$$T_{\underline{\tau}}^{(1)} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \hline \end{array} \quad T_{\underline{\tau}}^{(2)} = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \quad T_{\underline{\tau}}^{(3)} = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array}$$

**Lemma 3.4.4.** Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$  be a degree  $r$  head. If  $\underline{\tau} := (\tau_1, \dots, \tau_r), \underline{\gamma} := (\gamma_1, \dots, \gamma_r)$  are two elements in  $\text{WStd}_{\underline{\theta}}$ , then  $\lambda_{\underline{\tau}}^{(k)} = \lambda_{\underline{\gamma}}^{(k)} = \lambda_{\underline{\theta}}^{(k)}$  and  $\mu_{\underline{\tau}}^{(k)} = \mu_{\underline{\gamma}}^{(k)} = \mu_{\underline{\theta}}^{(k)}$ ,  $1 \leq k \leq d'_L$

*Proof.* By Proposition 3.1.11 we know that  $\text{Class}_{\theta_i} = \text{Class}_{\tau_i} = \text{Class}_{\gamma_i}$  for all  $1 \leq i \leq r$ . Thus for all  $1 \leq i \leq r$ ,  $1 \leq i \leq d$ , and  $1 \leq k \leq d'_L$  we have  $\theta_{i_j} \in \text{Block}_{L,k} \iff \tau_{i_j} \in \text{Block}_{L,k} \iff \gamma_{i_j} \in \text{Block}_{L,k}$ . As the shape of the associated tableaux depends only on the block membership of the entries we are done.  $\square$

**Definition 3.4.5.** Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$  be a degree  $r$  head. Set  $V_k := \mathbb{C}^{N_k}$ ,  $1 \leq k \leq d'_L$ . Define a vector space map on the basis elements of  $\langle \text{Std}_{\underline{\theta}} \rangle$  as follows:

$$\begin{aligned} \Psi_{\underline{\theta}} : \langle \text{Std}_{\underline{\theta}} \rangle &\longrightarrow \mathbb{W}_{\underline{\theta}} := \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}}(V_1) \otimes \dots \otimes \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(d'_L)}/\mu_{\underline{\theta}}^{(d'_L)}}(V_{d'_L}) \\ p_{\tau_1} \cdots p_{\tau_r} &\longmapsto T_{\underline{\tau}}^{(1)} \otimes \dots \otimes T_{\underline{\tau}}^{(d'_L)} \end{aligned}$$

where  $\underline{\tau} := (\tau_1, \dots, \tau_r)$ .

Lemma 3.4.4, the definition of the semistandard tableaux  $T_{\underline{\tau}}^{(k)}$  for  $1 \leq k \leq d'_L$ , and Theorem 2.4.1 give that this map is well defined and takes basis vectors to basis vectors.

Subsequently when we refer to the Weyl modules in the above tensor product we will write them as  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(k)}/\mu_{\underline{\theta}}^{(k)}}$ , omitting the  $(V_k)$ , so long as no confusion may arise from doing so.

**Proposition 3.4.6.** Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$  be a degree  $r$  head. The map  $\Psi_{\underline{\theta}}$  is a vector space isomorphism.

*Proof.* We describe a map  $\Phi_{\underline{\theta}}$  going from  $\mathbb{W}_{\underline{\theta}}$  to  $\langle \text{Std}_{\underline{\theta}} \rangle$ . Let  $T^{(1)} \otimes \dots \otimes T^{(d'_L)}$  be a basis vector of  $\mathbb{W}_{\underline{\theta}}$ . Then each  $T^{(k)}$  is a SSYT on the skew diagram  $\lambda_{\underline{\theta}}^{(k)}/\mu_{\underline{\theta}}^{(k)}$ . We now reverse the process described at the beginning of this section.

We first define the tableaux  $\tilde{T}^{(k)}$  by setting it equal to  $T^{(k)}$  with  $\hat{a}_k$  added to each entry. Then we define  $\tilde{T}^{(k)}$  to be the tableaux on  $\tilde{\lambda}_{\underline{\theta}}^{(k)}/\tilde{\mu}_{\underline{\theta}}^{(k)}$  that corresponds to adding the empty rows and columns to  $\tilde{T}^{(k)}$  that change its shape from  $\lambda_{\underline{\theta}}^{(k)}/\mu_{\underline{\theta}}^{(k)}$  to  $\tilde{\lambda}_{\underline{\theta}}^{(k)}/\tilde{\mu}_{\underline{\theta}}^{(k)}$ . Finally we combine  $\tilde{T}^{(1)}, \dots, \tilde{T}^{(d'_L)}$  into the square tableaux  $T$  of shape  $(r^d)$ , again in the reverse of the process described at the beginning of this section.

When comparing two boxes of  $T$  that are in the same block, the requirements for semistandardness are fulfilled since the individual tableaux associated with each block is a SSYT. When comparing two boxes of  $T$  that are not in the same block, if these two entries violated semistandardness then the same boxes in  $T_{\underline{\theta}}$  would violate semistandardness. Thus  $T$  is a SSYT.

Finally we define  $\underline{\tau} = (\tau_1, \dots, \tau_r)$  by letting the columns of  $T_{\underline{\tau}}$  correspond to the  $\tau_i$  in  $\underline{\tau}$ , but with their order reversed. The fact that  $T$  is semistandard implies that  $\underline{\tau}$  is standard, and the fact that  $T(i, j) \in \text{Block}_{L,k} \iff T_{\underline{\theta}}(i, j) \in \text{Block}_{L,k}$  for all  $k$  implies that  $\theta_{i_j} \in \text{Block}_{L,k} \iff \tau_{i_j} \in \text{Block}_{L,k}$  which implies that  $\theta_i = \theta_{\tau_i}$ . That is  $p_{\tau_1} \cdots p_{\tau_r} \in \langle \text{Std}_{\underline{\theta}} \rangle$ .

But then it is clear that the map  $\Phi_{\underline{\theta}}$  is well defined and in fact the inverse of  $\Psi_{\underline{\theta}}$ . This can be trivially verified on the basis vectors. And thus  $\Psi_{\underline{\theta}}$  is a vector space isomorphism.  $\square$

For  $p_{\tau_1} \cdots p_{\tau_r} \in \text{Std}_{\underline{\theta}}^{\geq str}$  let  $\overline{p_{\tau_1} \cdots p_{\tau_r}}$  denote its class in  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  under the canonical quotient map

$$\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle \rightarrow \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle.$$

Then  $\{\overline{p_{\tau_1} \cdots p_{\tau_r}}, (\tau_1, \dots, \tau_r) \in \text{WStd}_{\underline{\theta}}\}$  is a basis for  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$ .

Let  $\{(T_{\underline{\tau}}^{(1)} \otimes \dots \otimes T_{\underline{\tau}}^{(d'_L)})^*, \underline{\tau} \in \text{WStd}_{\underline{\theta}}\}$  be the basis of  $\mathbb{W}_{\underline{\theta}}^*$  dual to  $\{T_{\underline{\tau}}^{(1)} \otimes \dots \otimes T_{\underline{\tau}}^{(d'_L)}, \underline{\tau} \in \text{WStd}_{\underline{\theta}}\}$ .

**Definition 3.4.7.** The isomorphism  $\Psi_{\underline{\theta}}$  induces a vector space map

$$\begin{aligned} \overline{\Psi}_{\underline{\theta}} : \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle &\longrightarrow \mathbb{W}_{\underline{\theta}}^* \\ \overline{p_{\tau_1} \cdots p_{\tau_r}} &\longmapsto (T_{\underline{\tau}}^{(1)} \otimes \cdots \otimes T_{\underline{\tau}}^{(d'_L)})^* \end{aligned}$$

for  $\underline{\tau} = (\tau_1, \dots, \tau_r) \in \text{WStd}_{\underline{\theta}}$ .

**Proposition 3.4.8.** *The map  $\overline{\Psi}_{\underline{\theta}}$  is an isomorphism.*

*Proof.* That this map is an isomorphism follows from the fact that it is the composition of three isomorphisms. The first is from  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  to  $\langle \text{Std}_{\underline{\theta}} \rangle$  (cf. Remark 3.2.9), the second is from  $\langle \text{Std}_{\underline{\theta}} \rangle$  to  $W_{\underline{\theta}}$  (cf. Proposition 3.4.6), and the third is the canonical isomorphism from  $W_{\underline{\theta}}$  to  $W_{\underline{\theta}}^*$ .  $\square$

Our goal now is to show that  $\mathbb{W}_{\underline{\theta}}^*$  has a canonical  $L$ -module structure and then use the map  $\overline{\Psi}_{\underline{\theta}}$  to relate its  $L$ -module structure to the  $L$ -module structure of  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  (cf. Corollary 3.3.5).

**3.5. The  $\mathfrak{l}$ -module structure of  $\mathbb{W}_{\underline{\theta}}$  and the Implications for our Main Theorem.** Recall that  $L = \text{GL}_{N_1} \times \cdots \times \text{GL}_{N_{d'_L}}$  and

$$\mathbb{W}_{\underline{\theta}}^* = (\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}} \otimes \cdots \otimes \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(d'_L)}/\mu_{\underline{\theta}}^{(d'_L)}})^* \cong (\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}})^* \otimes \cdots \otimes (\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(d'_L)}/\mu_{\underline{\theta}}^{(d'_L)}})^*.$$

Each  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(i)}/\mu_{\underline{\theta}}^{(i)}}$  is a Weyl Module and thus has a canonical  $\text{GL}_{N_i}$ -module structure. Thus  $(\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(i)}/\mu_{\underline{\theta}}^{(i)}})^*$  has an induced  $\text{GL}_{N_i}$ -module structure. The  $L$ -module structure for  $\mathbb{W}_{\underline{\theta}}^*$  is simply given by the induced product structure.

Recall the following result due originally to Schur.

**Theorem 3.5.1.** *Let  $V$  and  $W$  be two finite dimensional polynomial  $\text{GL}_N$  representations. Then  $V$  and  $W$  are isomorphic if and only if  $\text{char}(V) = \text{char}(W)$ .*

*Proof.* For a proof of this refer to [Gre07, Theorem 3.5 and the second remark following the proof].  $\square$

In particular two polynomial  $L$  representations are isomorphic if and only if their characters are equal.

Let  $T_L \subset T$  be the maximal torus in  $L$ . In Proposition 3.4.8 we exhibited a vector space isomorphism from  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  to  $\mathbb{W}_{\underline{\theta}}^*$  that takes the  $T_L$  weight vector  $\overline{p_{\tau_1} \cdots p_{\tau_r}}$ ,  $\underline{\tau} = (\tau_1, \dots, \tau_r) \in \text{WStd}_{\underline{\theta}}$  to the  $T_L$  weight vector  $(T_{\underline{\tau}}^{(1)} \otimes \cdots \otimes T_{\underline{\tau}}^{(d'_L)})^*$ . We will use this to relate the characters of these two  $L$ -modules.

Since  $\mathbb{C}[X(w)]$  is a quotient of the polynomial  $\text{GL}_N$  representation  $\mathbb{C}[\mathbb{P}(\wedge^d \mathbb{C}^N)]$  by an  $L$ -stable ideal, it is a polynomial  $L$  representation, and thus any  $L$ -subrepresentation is a polynomial  $L$ -representation. Thus the quotient of two polynomial representations,  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  is a polynomial  $L$  representation.

Since the map  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle \rightarrow \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  is  $L$ -equivariant, and thus  $T_L$ -equivariant, we have that

$$wt(\overline{p_{\tau_1} \cdots p_{\tau_r}}) = wt(p_{\tau_1} \cdots p_{\tau_r}) = wt(p_{\tau_1}) + \cdots + wt(p_{\tau_r}).$$

As discussed in Remark 2.1.3 we have that the weight of  $p_{\tau}$  is given by the sequence  $\chi_{\tau} := (\chi_1, \dots, \chi_N)$  where



$$\chi_i := \begin{cases} 0 & i \in \tau \\ 1 & i \notin \tau \end{cases}$$

for all  $1 \leq i \leq N$ . Let  $n_{\underline{\tau}}^{(i)}$  equal the number times the value  $i$  appears in  $\underline{\tau}$ . Combining these results we have that  $wt(\overline{p_{\tau_1} \cdots p_{\tau_r}}) = (r - n_{\underline{\tau}}^{(1)}, \dots, r - n_{\underline{\tau}}^{(N)})$ . And thus

$$char(\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle) = \sum_{\underline{\tau} \in \text{WStd}_{\underline{\theta}}} e^{(r - n_{\underline{\tau}}^{(1)}, \dots, r - n_{\underline{\tau}}^{(N)})} \text{ (cf. Section 2.1)}$$

Regarding  $char(\mathbb{W}_{\underline{\theta}}^*)$  we have that

$$wt((T_{\underline{\tau}}^{(1)} \otimes \cdots \otimes T_{\underline{\tau}}^{(d'_L)})^*) = -wt(T_{\underline{\tau}}^{(1)} \otimes \cdots \otimes T_{\underline{\tau}}^{(d'_L)})$$

for all  $\tau \in \text{Std}_{\underline{\theta}}$ . Let  $\gamma_{\tau} := (\gamma_1, \dots, \gamma_N)$  be the weight of  $T_{\underline{\tau}}^{(k)}$ . Then  $\gamma_i$  is equal to the number of entries in  $T_{\underline{\tau}}^{(k)}$  equal to  $i - \widehat{a}_k$  for all  $i \in \{\widehat{a}_k + 1, \dots, \widehat{a}_{k+1}\}$  and zero otherwise. But  $n_{\underline{\tau}}^{(i)}$  is the number of entries in  $T_{\underline{\tau}}^{(k)}$  equal to  $i - \widehat{a}_k$ . Thus  $wt((T_{\underline{\tau}}^{(1)} \otimes \cdots \otimes T_{\underline{\tau}}^{(d'_L)})^*) = (-n_{\underline{\tau}}^{(1)}, \dots, -n_{\underline{\tau}}^{(N)})$  which implies

$$char(\mathbb{W}_{\underline{\theta}}^*) = \sum_{\underline{\tau} \in \text{WStd}_{\underline{\theta}}} e^{(-n_{\underline{\tau}}^{(1)}, \dots, -n_{\underline{\tau}}^{(N)})}.$$

So the characters of the  $L$ -modules  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  and  $\mathbb{W}_{\underline{\theta}}^*$  are not quite equal. The weight of each  $T_L$ -weight vector is off by  $(r, \dots, r)$ . This is remedied by tensoring  $\mathbb{W}_{\underline{\theta}}^*$  with the  $L$ -representation  $D_r := \det_{N_1}^r \otimes \cdots \otimes \det_{N_{d'_L}}^r$ , where the  $\det_{N_k}^r$  are the  $\text{GL}_{N_k}$ -modules in Definition 2.4.4.

**Proposition 3.5.2.** *The  $L$ -modules  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  and  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  are isomorphic.*

*Proof.* We have

$$char(\mathbb{W}_{\underline{\theta}}^* \otimes D_r) = \sum_{\underline{\tau} \in \text{WStd}_{\underline{\theta}}} e^{(-n_{\underline{\tau}}^{(1)}, \dots, -n_{\underline{\tau}}^{(N)})} e^{(r, \dots, r)} = \sum_{\underline{\tau} \in \text{WStd}_{\underline{\theta}}} e^{(r - n_{\underline{\tau}}^{(1)}, \dots, r - n_{\underline{\tau}}^{(N)})}$$

As we will see in Corollary 3.5.9 the representation  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  may be decomposed into a direct sum of polynomial irreducible  $L$ -representations and thus is itself a polynomial representation.

Thus since  $char(\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle) = char(\mathbb{W}_{\underline{\theta}}^* \otimes D_r)$  we have by Theorem 3.5.1 that  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  is isomorphic to  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  as  $L$ -modules.  $\square$

**Theorem 3.5.3.** *Let  $\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}$ . There exists a  $L$ -module  $U_{\underline{\theta}}$  such that we have the following  $L$ -module isomorphisms:*

- (a)  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle = U_{\underline{\theta}} \oplus \langle \text{Std}_{\underline{\theta}}^{> str} \rangle.$
- (b)  $\langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} := (\theta_1, \dots, \theta_r) \in \text{Head}_{L,r}} U_{\underline{\theta}}.$
- (c)  $U_{\underline{\theta}} \cong \mathbb{W}_{\underline{\theta}}^* \otimes D_r$

*Proof.* (a) We are in characteristic 0; so  $L$  being reductive we have that  $L$  is linearly reductive. Thus any  $L$ -module is completely reducible. This implies  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  is completely reducible, and since  $\langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  is a  $L$ -submodule of  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  it must have a  $L$ -module complement which we denote  $U_{\underline{\theta}}$ . Thus  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle = U_{\underline{\theta}} \oplus \langle \text{Std}_{\underline{\theta}}^{> str} \rangle$  as  $L$ -modules.

(b) We have the following vector space isomorphisms

$$(3.5.4) \quad \begin{aligned} U_{\underline{\theta}} &\cong \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle \quad (\text{by (a)}) \\ &\cong \langle \text{Std}_{\underline{\theta}} \rangle \quad (\text{Remark 3.2.9}) \end{aligned}$$

And by Corollary 3.2.6 we have that

$$(3.5.5) \quad \langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} \langle \text{Std}_{\underline{\theta}} \rangle$$

as vector spaces.

Now consider  $U_{\underline{\theta}_1}$  and  $U_{\underline{\theta}_2}$  for  $\underline{\theta}_1, \underline{\theta}_2 \in \text{Head}_{L,r}$  with  $\underline{\theta}_1 \neq \underline{\theta}_2$ . Then we claim that  $U_{\underline{\theta}_1} \cap U_{\underline{\theta}_2} = \{0\}$ . To see why this is the case we have two possibilities to consider.

**Case 1:**  $\underline{\theta}_1$  and  $\underline{\theta}_2$  comparable in the partial order  $\geq_{str}$ . Then without loss of generality say  $\underline{\theta}_1 >_{str} \underline{\theta}_2$ . Then  $\langle \text{Std}_{\underline{\theta}_2}^{\geq str} \rangle \subseteq \langle \text{Std}_{\underline{\theta}_1}^{\geq str} \rangle$ . But then since  $U_{\underline{\theta}_1}$  is an  $L$ -module complement of  $\langle \text{Std}_{\underline{\theta}_1}^{\geq str} \rangle$ , and  $U_{\underline{\theta}_1} \subset \langle \text{Std}_{\underline{\theta}_2}^{\geq str} \rangle$ , this implies  $U_{\underline{\theta}_1} \cap U_{\underline{\theta}_2} = \{0\}$ .

**Case 2:**  $\underline{\theta}_1$  and  $\underline{\theta}_2$  noncomparable in the partial order  $\geq_{str}$ . This implies

$$(3.5.6) \quad \text{Std}_{\underline{\theta}_2}^{\geq str} \cap \text{Std}_{\underline{\theta}_1} = \emptyset.$$

Let  $f \in U_{\underline{\theta}_1}$ , then since  $U_{\underline{\theta}_1} \subset \langle \text{Std}_{\underline{\theta}_1}^{\geq str} \rangle$  we have  $f = \sum a_i f_i$  for some  $a_i \in \mathbb{C}$  and  $f_i \in \text{Std}_{\underline{\theta}_1}^{\geq str}$ . Note that at least one of these  $f_i$  is in  $\text{Std}_{\underline{\theta}_1}$  and appears with nonzero  $a_i$ , otherwise  $f \in \langle \text{Std}_{\underline{\theta}_1}^{\geq str} \rangle$ , which contradicts the definition of  $U_{\underline{\theta}_1}$ . So we can rewrite  $f = \sum b_j g_j + \sum c_l h_l$  for some  $b_j, c_l \in \mathbb{C}$ ,  $g_j \in \text{Std}_{\underline{\theta}_1}$  and  $h_l \in \text{Std}_{\underline{\theta}_1}^{\geq str}$  with not all  $b_j$  equal to zero.

Now suppose  $f \in U_{\underline{\theta}_2}$ . This implies  $f = \sum d_i x_i$  for some  $d_i \in \mathbb{C}$  and  $x_i \in \text{Std}_{\underline{\theta}_2}^{\geq str}$ . Combining the two different expressions for  $f$  gives

$$(3.5.7) \quad \sum b_j g_j = \sum d_i x_i - \sum c_l h_l.$$

We have the  $g_j \in \text{Std}_{\underline{\theta}_1}$ ,  $x_i \in \text{Std}_{\underline{\theta}_2}^{\geq str}$ , and  $h_k \in \text{Std}_{\underline{\theta}_1}^{\geq str}$ . Now by (3.5.6) and the fact that

$$\text{Std}_{\underline{\theta}_1} \cap \text{Std}_{\underline{\theta}_1}^{\geq str} = \emptyset.$$

this means that in (3.5.7) we are writing a linear combination of standard monomials in  $\text{Std}_{\underline{\theta}_1}$  as a linear combination of standard monomials not in  $\text{Std}_{\underline{\theta}_1}$ , hence both sides of (3.5.7) should equal zero. Since not all the  $b_j$  are zero this is a violation of the linear independence of the standard monomials. Thus  $f \notin U_{\underline{\theta}_2}$  and we have  $U_{\underline{\theta}_1} \cap U_{\underline{\theta}_2} = \{0\}$ .

Thus the subspace of  $\langle \text{Std}_r \rangle$  that is defined as the sum of all the  $U_{\underline{\theta}}$ ,  $\underline{\theta} \in \text{Head}_{L,r}$ , is a direct sum, that is

$$\sum_{\underline{\theta} \in \text{Head}_{L,r}} U_{\underline{\theta}} = \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} U_{\underline{\theta}}.$$

But (3.5.4) and (3.5.5) imply that this subspace is in fact equal to  $\langle \text{Std}_r \rangle$  by dimension considerations. Thus we have

$$\langle \text{Std}_r \rangle = \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} U_{\underline{\theta}}$$

as vector spaces. Now since each  $U_{\underline{\theta}}$  is  $L$ -stable this is in fact an isomorphism of  $L$ -modules.

(c) We have that  $U_{\underline{\theta}} \cong \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  as  $L$ -modules. And by Proposition 3.5.2 we have  $\langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle / \langle \text{Std}_{\underline{\theta}}^{\geq str} \rangle$  is isomorphic to  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  as  $L$ -modules.  $\square$

**Remark 3.5.8.** When  $\theta \in \text{Head}_{L,1}$  ( $= \text{Head}_L$ ) we have that  $\langle \text{Std}_\theta \rangle$  is  $L$ -stable. This can be seen by noting that it will be  $\mathfrak{l}$ -stable, which follows immediately from the description of the  $\mathfrak{l}$ -action in Section 3.5 and Proposition 3.1.11. This implies that  $U_\theta \cong \langle \text{Std}_\theta \rangle$  as  $L$ -modules.

Further, when  $\theta \in \text{Head}_{L,1}$  we have that

$$\mathbb{W}_\theta = \mathbb{W}^{(1^{m_1})} \otimes \dots \otimes \mathbb{W}^{(1^{m_{d'_L}})}.$$

for some non-negative integers  $m_1, \dots, m_{d'_L}$  (cf. Section 3.4).

Thus for such  $\theta$  we have that  $U_\theta \cong \langle \text{Std}_\theta \rangle$  is an irreducible  $L$ -module.

**Corollary 3.5.9.** *The ring  $\mathbb{C}[X(w)]$  has the following decomposition into irreducible  $L$ -modules*

$$\bigoplus_{r \geq 1} \bigoplus_{\underline{\theta} \in \text{Head}_{L,r}} \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{((r^{N_1})/\nu_{\underline{\theta}}^{(1)})^\pi} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}^{\lambda_{\underline{\theta}}^{(1)}}} \right) \otimes \dots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_L)}} \left( \mathbb{W}^{((r^{N_{d'_L}})/\nu_{\underline{\theta}}^{(d'_L)})^\pi} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(d'_L)}, \nu_{\underline{\theta}}^{(d'_L)}}^{\lambda_{\underline{\theta}}^{(d'_L)}}} \right)$$

where for all  $1 \leq i \leq d'_L$  the  $\lambda_{\underline{\theta}}^{(i)}$ ,  $\mu_{\underline{\theta}}^{(i)}$  are the partitions defined in Section 3.4 and the innermost direct sums are over all partitions  $\nu_{\underline{\theta}}^{(i)}$  such that  $|\nu_{\underline{\theta}}^{(i)}| = |\lambda_{\underline{\theta}}^{(i)}| - |\mu_{\underline{\theta}}^{(i)}|$ .

*Proof.* The decomposition of  $\mathbb{W}_\theta$  into irreducible  $L$ -modules may be obtained by using 2.4.2;

$$\begin{aligned} \mathbb{W}_\theta &= \mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}} \otimes \dots \otimes \mathbb{W}^{\lambda_{\underline{\theta}}^{(d'_L)}/\mu_{\underline{\theta}}^{(d'_L)}} \\ &= \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(1)}} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}^{\lambda_{\underline{\theta}}^{(1)}}} \right) \otimes \dots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_L)}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(d'_L)}} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(d'_L)}, \nu_{\underline{\theta}}^{(d'_L)}}^{\lambda_{\underline{\theta}}^{(d'_L)}}} \right). \end{aligned}$$

And we may use the above decomposition to find

$$\begin{aligned} \mathbb{W}_\theta^* \otimes D_r &= \left( \mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}} \otimes \dots \otimes \mathbb{W}^{\lambda_{\underline{\theta}}^{(d'_L)}/\mu_{\underline{\theta}}^{(d'_L)}} \right)^* \otimes D_r \\ &= \left( \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(1)}} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}^{\lambda_{\underline{\theta}}^{(1)}}} \right) \otimes \dots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_L)}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(d'_L)}} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(d'_L)}, \nu_{\underline{\theta}}^{(d'_L)}}^{\lambda_{\underline{\theta}}^{(d'_L)}}} \right) \right)^* \otimes D_r \\ &= \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(1)}} \right)^* \otimes \det_{N_1}^r \right)^{\oplus c_{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}^{\lambda_{\underline{\theta}}^{(1)}}} \right) \otimes \dots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_L)}} \left( \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(d'_L)}} \right)^* \otimes \det_{N_{d'_L}}^r \right)^{\oplus c_{\mu_{\underline{\theta}}^{(d'_L)}, \nu_{\underline{\theta}}^{(d'_L)}}^{\lambda_{\underline{\theta}}^{(d'_L)}}} \right) \\ &= \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{((r^{N_1})/\nu_{\underline{\theta}}^{(1)})^\pi} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}^{\lambda_{\underline{\theta}}^{(1)}}} \right) \otimes \dots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_L)}} \left( \mathbb{W}^{((r^{N_{d'_L}})/\nu_{\underline{\theta}}^{(d'_L)})^\pi} \right)^{\oplus c_{\mu_{\underline{\theta}}^{(d'_L)}, \nu_{\underline{\theta}}^{(d'_L)}}^{\lambda_{\underline{\theta}}^{(d'_L)}}} \right) \text{ (by (2.4.5)).} \end{aligned}$$

As each  $((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi$  is a partition, this gives a decomposition of  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  into irreducible  $L$  modules. This also shows that  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  is a polynomial representation. Combining this decomposition of  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$  with Theorem 3.5.3 we get our desired result.  $\square$

**Remark 3.5.10.** Let  $L_w$  be the Levi subgroup of the stabilizer  $Q_w$ . If  $d'_{L_w} = 1$  we have that  $L_w = \mathrm{GL}_N$  and  $\mathbb{C}[X(w)]_r \cong \mathbb{W}^{(r^{N-d})}$  (cf. the proof of Theorem 4.0.17(a)). Further, Theorem 3.5.3 and Corollary 3.5.9 give the decomposition of  $\mathbb{C}[X(w)]_r$  for any  $L$  the Levi part of a parabolic subgroup  $Q \subset Q_w$ . This  $L$  is a subgroup of  $L_w$ , in fact  $L = \mathrm{GL}_{N_1} \times \cdots \times \mathrm{GL}_{N_{d'_L}}$  is embedded diagonally in  $L_w = \mathrm{GL}_N$ . Further  $X(w) = G_{d,N}$ . Thus as a consequence of our explicit decomposition we get the branching rules for the Weyl module  $\mathbb{W}^{(r^{N-d})}$  for any  $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_{d'_L}}$  with  $m_1 + \cdots + m_{d'_L} = N$  diagonally embedded in  $\mathrm{GL}_N$ . This branching rule is discussed in much greater generality in [HTW05] for  $\mathrm{GL}_m \times \mathrm{GL}_n$  embedded diagonally in  $\mathrm{GL}_{n+m}$ . It seems reasonable to expect that further exploration of the cases when  $d'_{L_w} > 1$  might yield additional non-trivial branching rules for polynomial representations of  $\mathrm{GL}_N$ .

#### 4. MULTIPLICITY CONSEQUENCES OF THE DECOMPOSITION

In this section we use the decomposition given in Theorem 3.5.3 to classify the multiplicity free Schubert varieties, that is those Schubert varieties whose homogeneous coordinate rings have a decomposition into irreducible  $L$ -modules that is multiplicity free. To make the classification statement more tractable we will restrict our consideration to the case when  $L$  is the Levi subgroup of the stabilizer itself, in the notation of Section 3.1 when  $Q = Q_w$ . All of the methods we develop are extendable with some care to the more general case where  $L$  is the Levi subgroup of some parabolic subgroup of the stabilizer.

For this section fix  $d, N$  positive integers with  $d < N$ . Let  $P = P_{\hat{d}}$  and  $w \in W^P$ . Let  $L_w$  be the Levi subgroup of the stabilizer  $Q_w$  of  $X(w)$ , and  $\mathfrak{l}_w = \mathrm{Lie}(L_w)$  its associated Lie Algebra.

**Proposition 4.0.1.** *The decomposition of the ring  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if the decomposition of*

$$\bigoplus_{r \geq 1} \bigoplus_{\underline{\theta} \in \mathrm{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}$$

*into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* Let  $\underline{\theta} \in \mathrm{Head}_{L_w, r}$ . As in the proof of Corollary 3.5.9 we use the fact that

$$\mathbb{W}_{\underline{\theta}} = \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(1)}} \right)^{\oplus c \frac{\lambda_{\underline{\theta}}^{(1)}}{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}} \right) \otimes \cdots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_{L_w})}} \left( \mathbb{W}^{\nu_{\underline{\theta}}^{(d'_{L_w})}} \right)^{\oplus c \frac{\lambda_{\underline{\theta}}^{(d'_{L_w})}}{\mu_{\underline{\theta}}^{(d'_{L_w})}, \nu_{\underline{\theta}}^{(d'_{L_w})}}} \right)$$

to conclude that

$$\mathbb{W}_{\underline{\theta}}^* \otimes D_r = \left( \bigoplus_{\nu_{\underline{\theta}}^{(1)}} \left( \mathbb{W}^{((r^{N_1})/\nu_{\underline{\theta}}^{(1)})^\pi} \right)^{\oplus c \frac{\lambda_{\underline{\theta}}^{(1)}}{\mu_{\underline{\theta}}^{(1)}, \nu_{\underline{\theta}}^{(1)}}} \right) \otimes \cdots \otimes \left( \bigoplus_{\nu_{\underline{\theta}}^{(d'_{L_w})}} \left( \mathbb{W}^{((r^{N_{d'_{L_w}}})/\nu_{\underline{\theta}}^{(d'_{L_w})})^\pi} \right)^{\oplus c \frac{\lambda_{\underline{\theta}}^{(d'_{L_w})}}{\mu_{\underline{\theta}}^{(d'_{L_w})}, \nu_{\underline{\theta}}^{(d'_{L_w})}}} \right).$$

These are decompositions into irreducible  $L_w$ -modules. For a fixed  $r$ , a partition uniquely defines its  $r^{N_k}$ -complement (note that  $N_1, \dots, N_{d'_{L_w}}$  are all fixed once we choose a Levi subgroup), and hence uniquely defines its  $\pi$ -rotation. And thus for a fixed  $r$ ,

$$\bigoplus_{\underline{\theta} \in \text{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}^* \otimes D_r$$

has a multiplicity free decomposition if and only if

$$\bigoplus_{\underline{\theta} \in \text{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}$$

has a multiplicity free decomposition.

Now let  $r, r' \geq 1$  with  $r' > r$  and let  $\underline{\theta} \in \text{Head}_{L_w, r}$ ,  $\underline{\theta}' \in \text{Head}_{L_w, r'}$ . Further, let  $I$  be an irreducible  $L_w$ -submodule of  $\mathbb{W}_{\underline{\theta}}^* \otimes D_r$ . Then as we have seen above  $I$  is of the form  $\mathbb{W}^{((r^{N_1})/\nu_{\underline{\theta}}^{(1)})^\pi} \otimes \dots \otimes \mathbb{W}^{((r^{N_{d'_{L_w}}})/\nu_{\underline{\theta}}^{(d'_{L_w})})^\pi}$ . Recall each  $((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi$  is a partition and  $\nu_{\underline{\theta}}^{(k)}$  is a partition such that  $\mathbb{W}_{\underline{\theta}}^{\nu_{\underline{\theta}}^{(k)}}$  is a  $\text{GL}_{N_k}$ -submodule of  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(k)}/\mu_{\underline{\theta}}^{(k)}}$ . But this implies that  $|\nu_{\underline{\theta}}^{(k)}| = |\lambda_{\underline{\theta}}^{(k)}/\mu_{\underline{\theta}}^{(k)}| = |\lambda_{\underline{\theta}}^{(k)}| - |\mu_{\underline{\theta}}^{(k)}|$ . From this we conclude that

$$(4.0.2) \quad |((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi| = rN_k - |\nu_{\underline{\theta}}^{(k)}| = rN_k - (|\lambda_{\underline{\theta}}^{(k)}| - |\mu_{\underline{\theta}}^{(k)}|)$$

Similarly let  $I'$  be an irreducible  $L_w$ -submodule of  $\mathbb{W}_{\underline{\theta}'}^* \otimes D_{r'}$ . By the same argument as above  $I'$  is of the form  $\mathbb{W}^{((r^{N_1})/\nu_{\underline{\theta}'}^{(1)})^\pi} \otimes \dots \otimes \mathbb{W}^{((r^{N_{d'_{L_w}}})/\nu_{\underline{\theta}'}^{(d'_{L_w})})^\pi}$  and we have

$$(4.0.3) \quad |((r^{N_k})/\nu_{\underline{\theta}'}^{(k)})^\pi| = r'N_k - |\nu_{\underline{\theta}'}^{(k)}| = r'N_k - (|\lambda_{\underline{\theta}'}^{(k)}| - |\mu_{\underline{\theta}'}^{(k)}|)$$

Now suppose that  $I \cong I'$  as  $L_w$ -modules. This would imply that  $\mathbb{W}^{((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi} \cong \mathbb{W}^{((r^{N_k})/\nu_{\underline{\theta}'}^{(k)})^\pi}$  as  $\text{GL}_{N_k}$ -modules for all  $1 \leq k \leq d'_{L_w}$ . In particular this implies that  $((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi = ((r^{N_k})/\nu_{\underline{\theta}'}^{(k)})^\pi$  which implies that they have the same number of blocks, that is  $|((r^{N_k})/\nu_{\underline{\theta}}^{(k)})^\pi| = |((r^{N_k})/\nu_{\underline{\theta}'}^{(k)})^\pi|$ . Thus we have

$$|((r^{N_1})/\nu_{\underline{\theta}}^{(1)})^\pi| + \dots + |((r^{N_{d'_{L_w}}})/\nu_{\underline{\theta}}^{(d'_{L_w})})^\pi| = |((r^{N_1})/\nu_{\underline{\theta}'}^{(1)})^\pi| + \dots + |((r^{N_{d'_{L_w}}})/\nu_{\underline{\theta}'}^{(d'_{L_w})})^\pi|.$$

The left hand side is

$$\begin{aligned} &= rN_1 - (|\lambda_{\underline{\theta}}^{(1)}| - |\mu_{\underline{\theta}}^{(1)}|) + \dots + rN_{d'_{L_w}} - (|\lambda_{\underline{\theta}}^{(d'_{L_w})}| - |\mu_{\underline{\theta}}^{(d'_{L_w})}|) \quad (\text{by (4.0.2)}) \\ &= rN_1 + \dots + rN_{d'_{L_w}} - rd \quad (\text{by (3.4.2)}) \end{aligned}$$

and the right hand side is

$$\begin{aligned} &= r'N_1 - (|\lambda_{\underline{\theta}'}^{(1)}| - |\mu_{\underline{\theta}'}^{(1)}|) + \dots + r'N_{d'_{L_w}} - (|\lambda_{\underline{\theta}'}^{(d'_{L_w})}| - |\mu_{\underline{\theta}'}^{(d'_{L_w})}|) \quad (\text{by (4.0.3)}) \\ &= r'N_1 + \dots + r'N_{d'_{L_w}} - r'd \quad (\text{by (3.4.2)}) \end{aligned}$$

Combining, we have that

$$rN_1 + \dots + rN_{d'_{L_w}} - rd = r'N_1 + \dots + r'N_{d'_{L_w}} - r'd$$

which means that  $(r' - r)(N_1 + \cdots + N_{d'_{L_w}}) = (r' - r)d$ . Canceling  $(r' - r)$ , since it is not zero, we get that  $N = N_1 + \cdots + N_{d'_{L_w}} = d$ , and this is a contradiction since  $d < N$ . Thus we can not have  $I \cong I'$ , which implies that we may not have isomorphisms between irreducible  $L_w$ -submodules in different degrees of

$$\bigoplus_{r \geq 1} \bigoplus_{\underline{\theta} \in \text{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}^* \otimes D_r.$$

And by a similar argument we may show that no such isomorphisms may occur between irreducible  $L_w$ -submodules in different degrees of

$$\bigoplus_{r \geq 1} \bigoplus_{\underline{\theta} \in \text{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}.$$

Thus we have shown that our desired multiplicity result holds for a fixed degree, and since no isomorphisms between  $L_w$ -submodules from different degrees can occur we are done.  $\square$

**Corollary 4.0.4.** *Let  $X(w)$  be a Schubert variety in  $G_{d,N}$ . The decomposition of the ring  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if the following two criteria are satisfied for all  $r \geq 1$ .*

*M1 For all  $\theta \in \text{Head}_{L_w, r}$  the  $\text{GL}_{N_i}$  module  $\mathbb{W}_{\theta}^{\lambda^{(i)}/\mu_{\theta}^{(i)}}(V_i)$  is multiplicity free for all  $1 \leq i \leq d'_{L_w}$ .*

*M2 For all  $\theta, \theta' \in \text{Head}_{L_w, r}$ , if  $\mathbb{W}_{\theta}^{\nu_{\theta}^{(1)}}(V_1) \otimes \cdots \otimes \mathbb{W}_{\theta}^{\nu_{\theta}^{(d'_{L_w})}}(V_{d'_{L_w}})$  is an irreducible  $L_w$  submodule*

*of  $\mathbb{W}_{\theta}$  and  $\mathbb{W}_{\theta'}^{\nu_{\theta'}^{(1)}}(V_1) \otimes \cdots \otimes \mathbb{W}_{\theta'}^{\nu_{\theta'}^{(d'_{L_w})}}(V_{d'_{L_w}})$  is an irreducible  $L_w$  submodule of  $\mathbb{W}_{\theta'}$ , then*

*$\mathbb{W}_{\theta}^{\nu_{\theta}^{(1)}}(V_1) \otimes \cdots \otimes \mathbb{W}_{\theta}^{\nu_{\theta}^{(d'_{L_w})}}(V_{d'_{L_w}}) \cong \mathbb{W}_{\theta'}^{\nu_{\theta'}^{(1)}}(V_1) \otimes \cdots \otimes \mathbb{W}_{\theta'}^{\nu_{\theta'}^{(d'_{L_w})}}(V_{d'_{L_w}})$  as  $L_w$  modules implies  $\theta = \theta'$ .*

*Proof.* If the decomposition of the ring  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free then clearly both  $M1$  and  $M2$  hold.

If both  $M1$  and  $M2$  hold for all  $r \geq 1$ , then we have that the decomposition of

$$\bigoplus_{r \geq 1} \bigoplus_{\underline{\theta} \in \text{Head}_{L_w, r}} \mathbb{W}_{\underline{\theta}}.$$

into irreducible  $L_w$ -modules is multiplicity free since, as we saw in the proof of Proposition 4.0.1, we may not have any isomorphisms between irreducible  $L_w$ -submodules in different degrees of this direct sum. By Proposition 4.0.1 we have that the above direct sum is multiplicity free if and only if  $\mathbb{C}[X(w)]$  is multiplicity free.  $\square$

We will say that a particular Schubert variety satisfies criterion  $M1$  if for all  $r \geq 1$  the first condition from Corollary 4.0.4 holds. And similarly for satisfying criterion  $M2$ .

These complexity of the classification can be greatly reduced if we impose some restrictions on  $w$ . In Proposition 4.0.5 we show that the decomposition of  $\mathbb{C}[X(w)]$  for a general  $w := (i_1, \dots, i_d) \in W^P$  may be written in terms of the decomposition of the homogeneous coordinate ring of a Schubert variety  $X(\overline{w})$  in a smaller Grassmannian, where  $\overline{w}$  satisfies certain restrictions(cf. Proposition 4.0.5(c)). In particular, the decomposition of  $\mathbb{C}[X(w)]$  will be multiplicity free if and only if the decomposition of  $\mathbb{C}[X(\overline{w})]$  is multiplicity free.



**Proposition 4.0.5.** *The element  $w$  is of the form  $(1, \dots, p, i_1, \dots, i_{\bar{d}})$  for some unique  $0 \leq p \leq d$  and  $\bar{d} \leq d$  with  $i_1 \neq p+1$ . Let  $\bar{P} = P_{\bar{d}} \subset \mathrm{GL}_{\bar{N}}$  where  $\bar{N} = i_{\bar{d}} - p$  and define  $\bar{w} := (i_1 - p, \dots, i_{\bar{d}} - p) \in W^{\bar{P}}$ . Let  $L_{\bar{w}}$  be the Levi subgroup of the stabilizer  $Q_{\bar{w}}$  of  $X(\bar{w})$*

- (a) *The Schubert variety  $X(w) \cong X(\bar{w}) \subseteq G_{\bar{d}, \bar{N}}$  as varieties. In addition  $\mathbb{C}[X(w)] \cong \mathbb{C}[X(\bar{w})]$  as  $\mathbb{C}$ -algebras.*
- (b) *The ring  $\mathbb{C}[X(w)]$  has a multiplicity free decomposition into irreducible  $L_w$ -modules if and only if  $\mathbb{C}[X(\bar{w})]$  has a multiplicity free decomposition into irreducible  $L_{\bar{w}}$ -modules.*
- (c) *The first entry of  $\bar{w}$  is not equal to 1 and the last entry equals  $\bar{N}$ .*

*Proof.* (a) We have the following diagram where  $I$ ,  $II$ , and  $III$  are the Plücker embeddings of the respective Grassmannians.

$$\begin{array}{ccccc}
 X(\bar{w}) & \xrightarrow{\sim} & X(w) & \xrightarrow{\sim} & X(w) \\
 \downarrow & & \downarrow & & \downarrow \\
 G_{\bar{d}, \bar{N}} & \hookrightarrow & G_{d, i_{\bar{d}}} & \hookrightarrow & G_{d, N} \\
 \downarrow I & & \downarrow II & & \downarrow III \\
 \mathbb{P}(\wedge^{\bar{d}} \mathbb{C}^{\bar{N}}) & \hookrightarrow & \mathbb{P}(\wedge^{\bar{d}} \mathbb{C}^{i_{\bar{d}}}) & \hookrightarrow & \mathbb{P}(\wedge^{\bar{d}} \mathbb{C}^N)
 \end{array}$$

The Hasse diagrams associated with  $X(w)$  and  $X(\bar{w})$  are identical. Also  $X(w)$  is cut out scheme theoretically from  $G_{d, N}$  by linear subspaces:  $\{p_{\tau}, \tau \not\leq w\}$  and  $X(\bar{w})$  is cut out scheme theoretically from  $G_{\bar{d}, \bar{N}}$  by linear subspaces:  $\{p_{\bar{\tau}}, \bar{\tau} \not\leq \bar{w}\}$ . From these it follows that we have a natural isomorphism of  $\mathbb{C}$ -algebras:  $\mathbb{C}[X(\bar{w})] \cong \mathbb{C}[X(w)]$ , in particular we have  $X(\bar{w}) \cong X(w)$ .

(b) As we saw in the proof of Proposition 4.0.1, we may not have isomorphisms between irreducible  $L_w$  (respectively  $L_{\bar{w}}$ )-submodules in different degrees of  $\mathbb{C}[X(w)]$  (respectively  $\mathbb{C}[X(\bar{w})]$ ). Hence, it suffices to show that for any  $r \geq 1$  we have that  $\mathbb{C}[X(w)]_r$  has a multiplicity free decomposition into irreducible  $L_w$ -modules if and only if  $\mathbb{C}[X(\bar{w})]_r$  has a multiplicity free decomposition into irreducible  $L_{\bar{w}}$ -modules.

Further, we will prove the result for  $p > 0$  and  $i_{\bar{d}} < N$ , as the cases when  $p = 0$  or  $i_{\bar{d}} = N$  are simpler versions of this general case. We have that

$$\begin{aligned}
 R'_{Q_w} &= \{n \in \{1, \dots, N-1\} \mid \exists k \text{ with } n = i_k \text{ and } i_k + 1 \neq i_{k+1}\} \\
 R'_{Q_{\bar{w}}} &= \left\{n \in \{1, \dots, \bar{N}-1\} \mid \exists k \text{ with } n = i_k - p \text{ and } (i_k - p) + 1 \neq i_{k+1} - p\right\}.
 \end{aligned}$$

Thus

$$R'_{Q_w} = \{p, b_1, \dots, b_s, i_{\bar{d}}\}$$

with  $b_1, \dots, b_s \in \{p+2, \dots, i_{\bar{d}}-2\}$ . In terms of the variables above

$$R'_{Q_{\bar{w}}} = \{b_1 - p, \dots, b_s - p\}.$$

This implies a relationship between the blocks of  $L_w$  (cf. Definition 3.1.2) and the blocks of  $L_{\bar{w}}$ :

$$(4.0.6) \quad \text{Block}_{L_{\bar{w}}, k} = \left\{n \in \{1, \dots, \bar{N}\} \mid n + p \in \text{Block}_{L_w, k+1}\right\} \text{ for } 1 \leq k \leq s$$

and that

$$(4.0.7) \quad L_w = \mathrm{GL}_p \times L_{\bar{w}} \times \mathrm{GL}_{N-i_{\bar{d}}}.$$

With this notation set we now proceed to the first step of the proof, showing a bijection between  $\text{Head}_{L_w, r}$  and  $\text{Head}_{L_{\overline{w}}, r}$ .

We have already remarked in part (a) that the Hasse diagrams associated to  $X(w)$  and  $X(\overline{w})$  are identical. We will need to make this more explicit. Recall that  $H_w := \{\tau \in W^P \mid \tau \leq w\}$  and thus all  $\tau \in H_w$  are of the form  $(1, \dots, p, j_1, \dots, j_{\overline{d}})$  with  $j_{\overline{d}} \leq i_{\overline{d}}$ . Define the map

$$\begin{aligned} \iota : H_w &\longrightarrow H_{\overline{w}} \\ (1, \dots, p, j_1, \dots, j_{\overline{d}}) &\longmapsto (j_1 - p, \dots, j_{\overline{d}} - p). \end{aligned}$$

This map is a bijection, and  $\tau_1 \leq \tau_2$  if and only if  $\iota(\tau_1) \leq \iota(\tau_2)$ . Thus the Hasse diagram  $\mathcal{H}_w$  is identical to the Hasse diagram  $\mathcal{H}_{\overline{w}}$ . In Section 3.1 we labeled the edge of the Hasse diagram  $\mathcal{H}_w$  connecting  $\tau_1$  and  $\tau_2$  by the unique  $s_{\alpha_m}$  such that  $\tau_1 = s_{\alpha_m} \tau_2$ . If  $\tau_1 = s_{\alpha_m} \tau_2$ , then  $\iota(\tau_1) = s_{\alpha_{m-p}} \iota(\tau_2)$ . Thus we see that if  $\tau_1$  is connected to  $\tau_2$  by an edge labeled by  $s_{\alpha_m}$  for  $m \in R'_{Q_w}$ , then  $\iota(\tau_1)$  is connected to  $\iota(\tau_2)$  by an edge labeled by  $s_{\alpha_{m-p}}$  for  $m - p \in R'_{Q_{\overline{w}}}$ . Further, no edges in  $\mathcal{H}_w$  are labeled by  $\alpha_p$  or  $\alpha_{i_{\overline{d}}}$  since all  $\tau \in H_w$  are of the form  $(1, \dots, p, j_1, \dots, j_{\overline{d}})$  with  $j_{\overline{d}} \leq i_{\overline{d}}$ .

This implies that the disjoint Hasse diagrams  $\widehat{\mathcal{H}}_w$  (cf. Proposition 3.1.11) and  $\widehat{\mathcal{H}}_{\overline{w}}$  are identical. Thus by Proposition 3.1.11 we have that  $\iota$  gives a bijection from  $\text{Head}_{L_w}$  to  $\text{Head}_{L_{\overline{w}}}$ . And this, combined with the fact that the Hasse diagram itself is identical implies that

$$\begin{aligned} \iota^{(r)} : \text{Head}_{L_w, r} &\longrightarrow \text{Head}_{L_{\overline{w}}, r} \\ (\theta_1, \dots, \theta_r) &\longmapsto (\iota(\theta_1), \dots, \iota(\theta_r)) \end{aligned}$$

is a bijection.

Our second step is to compare  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\iota^{(r)}(\underline{\theta})}$  for  $\underline{\theta} \in \text{Head}_{L_w, r}$ . The semistandard tableaux  $T_{\underline{\theta}}$  and  $T_{\iota^{(r)}(\underline{\theta})}$  are on the rectangular diagrams  $(d^r)$  and  $(\overline{d}^r) = ((d-p)^r)$ , respectively. The first  $p$  rows of  $T_{\underline{\theta}}$  always contain the values  $1, \dots, p$ . Then  $(i, j)$ th box of  $T_{\iota^{(r)}(\underline{\theta})}$  is equal to the  $(i+p, j)$ th box of  $T_{\underline{\theta}}$  minus  $p$ . In particular, in light of (4.0.6), this implies that the  $(i, j)$ th box of  $T_{\iota^{(r)}(\underline{\theta})}$  is in  $\text{Block}_{L_{\overline{w}}, k}$  if and only if the  $(i+p, j)$ th box of  $T_{\underline{\theta}}$  is in  $\text{Block}_{L_w, k+1}$ . Finally, note that there are no boxes in  $T_{\underline{\theta}}$  that contain entries from  $i_{\overline{d}} + 1$  to  $N$ . Combining these results gives us that

$$(4.0.8) \quad \mathbb{W}_{\underline{\theta}} = \mathbb{W}^{(p^r)} \otimes \mathbb{W}_{\iota^{(r)}(\underline{\theta})} \otimes \mathbb{W}^{(0)}.$$

As discussed in the proof of Corollary 4.0.4 we know that  $C[X(w)]_r$  has a multiplicity free decomposition into irreducible  $L_w$ -modules if and only if criterion  $M1$  and  $M2$  hold with respect to  $L_w$  for all  $\underline{\theta} \in \text{Head}_{L_w, r}$  and  $\mathbb{W}_{\underline{\theta}}$ . By (4.0.7), (4.0.8), and the bijection  $\iota^{(r)}$  this is equivalent to criterion  $M1$  and  $M2$  holding with respect to  $L_{\overline{w}}$  for all  $\underline{\theta}' \in \text{Head}_{L_{\overline{w}}, r}$  and  $\mathbb{W}_{\underline{\theta}'}$ . But this is equivalent to  $C[X(\overline{w})]_r$  having a multiplicity free decomposition into irreducible  $L_{\overline{w}}$ -modules.

(c) Since  $i_1 \neq p+1$  the first entry in  $\overline{w}$  is not equal to 1. The last entry in  $\overline{w}$  is  $i_{\overline{d}} - p = \overline{N}$ . □

In light of Proposition 4.0.5 we now restrict our consideration to those  $w := (i_1, \dots, i_d) \in W^P$  such that  $i_1 \neq 1$  and  $i_d = N$ .

**Lemma 4.0.9.** *The number of blocks of  $L_w$  is less than or equal  $d$ , that is  $d'_{L_w} \leq d$ .*

*Proof.* By the definition of  $R'_{Q_w}$  given in Proposition 3.1.1 we see that the only possible values in  $R'_{Q_w}$  are those entries in  $w$  not equal to  $N$ . As  $i_d = N$  there are at most  $d-1$  such entries. Thus  $|R'_{Q_w}| \leq d-1$ . But then  $d'_{L_w} = |R'_{Q_w}| + 1 \leq d$ . □

For a fixed  $w = (i_1, \dots, i_d)$  with  $i_1 \neq 1$  and  $i_d = N$  let

$$(4.0.10) \quad h_k = \#\{i \mid w_i \in \text{Block}_{L_w, k}\}$$

Then  $N_k \geq h_k$  for all  $1 \leq k \leq d'_{Lw}$  since  $w$  has no repeated values. However we may refine this bound. We know that  $h_k = N_k$  only when  $\hat{a}_k + 1, \dots, \hat{a}_{k+1}$  are all entries in  $w$ . But for all  $k > 1$  we have that  $\hat{a}_k \in R'_{Qw}$  which implies, by Proposition 3.1.1, that there exists an index  $m$  with  $i_m = \hat{a}_k$  and  $i_{m+1} \neq \hat{a}_k + 1$ . That is there is no entry of  $w$  equal to  $\hat{a}_k + 1$ . Thus  $h_k \neq N_k$  for  $k > 1$ . We conclude that for all  $k > 1$  we have  $N_k > h_k$ . In the case when  $k = 1$ , we know that 1 is not an entry of  $w$  and thus  $N_1 > h_1$ . Proposition 3.1.1 and the definition of the blocks (cf. Definition 3.1.2) imply that  $h_k \geq 1$  for  $1 \leq k < d'_{Lw}$ ; since  $i_d = N$  we have  $h_k \geq 1$  for  $k = d'_{Lw}$ . Combining these results we have that

$$(4.0.11) \quad N_k > h_k \geq 1 \text{ for all } 1 \leq k \leq d'_{Lw}.$$

In addition, since the blocks partition the set  $\{1, \dots, N\}$ , the  $h_k$  count the number of entries in  $w$ , thus

$$(4.0.12) \quad d = h_1 + h_2 + \dots + h_{d'_{Lw}}.$$

The final ingredient we will need for our classification theorem is an alternate method for indexing the set of degree 1 heads for a fixed  $w$ . This alternate notation will prove useful when generating examples that are not multiplicity free. For nonnegative integers  $m_1, \dots, m_{d'_{Lw}}$ , let  $\Theta(m_1, \dots, m_{d'_{Lw}})$  correspond to the sequence

$$(\hat{a}_2 - m_1 + 1, \dots, \hat{a}_2, \hat{a}_3 - m_2 + 1, \dots, \hat{a}_3, \dots, \hat{a}_{d'_{Lw}+1} - m_{d'_{Lw}} + 1, \dots, \hat{a}_{d'_{Lw}+1}).$$

In general this is a sequence of length  $m_1 + \dots + m_{d'_{Lw}}$  and may not even be an element of  $W^P$ . However if certain conditions are satisfied this will be a head of type  $L_w$ .

**Example 4.0.13.** As in Example 3.1.7 let  $d = 3$  and  $N = 9$ . Consider  $w = (3, 6, 9) \in W^{\widehat{P}_3}$ . Then  $X(w)$  is a Schubert variety in  $G_{3,9}$ . In this case  $R'_{Qw} = \{3, 6\}$  and  $R_{Qw} = \{1, 2, 4, 5, 7, 8\}$ . Choose  $R_Q := R_{Qw}$  for the parabolic subgroup  $Q = P_{R_Q}$ . Then  $\hat{a} := (\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) = \{0, 3, 6, 9\}$ . So  $\text{Block}_{Lw,1} = (1, 2, 3)$ ,  $\text{Block}_{Lw,2} = (4, 5, 6)$ , and  $\text{Block}_{Lw,3} = (7, 8, 9)$ .

Then

$$\Theta(3, 4, 1) = (1, 2, 3, 3, 4, 5, 6, 9)$$

$$\Theta(1, 0, 2) = (3, 8, 9)$$

$$\Theta(2, 1, 0) = (2, 3, 6)$$

and we see that only  $\Theta(1, 0, 2), \Theta(2, 1, 0) \in W^{\widehat{P}_3}$ . However, only  $\Theta(2, 1, 0) \in \text{Head}_{Lw}$  since  $(3, 8, 9) \not\leq w$ .

**Lemma 4.0.14.** *For  $m_1, \dots, m_{d'_{Lw}}$  nonnegative integers, the sequence  $\Theta(m_1, \dots, m_{d'_{Lw}})$  will be a head of type  $L_w$  (cf. Proposition 3.1.5), if and only if the following conditions on  $m_1, \dots, m_{d'_{Lw}}$  are all satisfied.*

- (1)  $m_1 + \dots + m_{d'_{Lw}} = d$
- (2)  $m_k \leq N_k$  for  $1 \leq k \leq d'_{Lw}$
- (3)  $m_1 + \dots + m_k \geq h_1 + \dots + h_k$  for  $1 \leq k < d'_{Lw}$

*Proof.* The sequence  $\Theta(m_1, \dots, m_{d'_{Lw}})$  has length  $d$  if and only if  $m_1 + \dots + m_{d'_{Lw}} = d$ .

If  $m_k \leq N_k = \hat{a}_{k+1} - \hat{a}_k$  then  $\hat{a}_k \leq \hat{a}_{k+1} - m_k < \hat{a}_{k+1} - m_k + 1$ . Thus  $\Theta(m_1, \dots, m_{d'_{Lw}})$  will have no repeated values. If  $m_k \not\leq N_k = \hat{a}_{k+1} - \hat{a}_k$  then  $\hat{a}_k > \hat{a}_{k+1} - m_k$  which implies  $\hat{a}_k \geq \hat{a}_{k+1} - m_k + 1$ . Thus  $\Theta(m_1, \dots, m_{d'_{Lw}})$  will have repeated values. So  $\Theta(m_1, \dots, m_{d'_{Lw}})$  has no repeated values if and only if  $m_k \leq N_k$  for  $1 \leq k \leq d'_{Lw}$ .

Thus conditions 1 and 2 are satisfied if and only if  $\Theta(m_1, \dots, m_{d'_{Lw}}) \in W^P$ .

We can realize  $w$  as  $\Theta(h_1, \dots, h_{d'_{L_w}})$ , and from this it is trivial to verify that  $\Theta(m_1, \dots, m_{d'_{L_w}}) \leq \Theta(h_1, \dots, h_{d'_{L_w}}) = w$  if and only if  $m_1 + \dots + m_k \geq h_1 + \dots + h_k$  for  $1 \leq k < d'_{L_w}$ . Thus all 3 conditions are satisfied if and only if  $\Theta(m_1, \dots, m_{d'_{L_w}}) \in H_w$ .

It is easy to check that a  $\Theta(m_1, \dots, m_{d'_{L_w}})$  satisfying these conditions also satisfies the third condition in Proposition 3.1.5, making it a head of type  $L_w$ .  $\square$

**Remark 4.0.15.** This method of indexing the heads is useful because it makes apparent the number of entries in each block. Consider a degree 1 head  $\Theta(m_1, \dots, m_{d'_{L_w}}) \in \text{Head}_{L_w,1}$ , we then have that

$$\mathbb{W}_{\Theta(m_1, \dots, m_{d'_{L_w}})} = \mathbb{W}^{(1^{m_1})} \otimes \dots \otimes \mathbb{W}^{(1^{m_{d'_{L_w}}})}.$$

**Example 4.0.16.** This method of indexing is also useful for understanding the structure of  $\mathbb{W}_{\underline{\theta}}$  when  $\underline{\theta} \in \text{Head}_{L_w,r}$ . Once again let us work with  $w = (3, 6, 9)$  and  $R_Q = R_{Q_w} = \{1, 2, 4, 5, 7, 8\}$  as in Example 4.0.13. Let  $\underline{\theta} = (\theta_1, \theta_2, \theta_3)$  where

$$\theta_1 = \Theta(1, 1, 1) = (3, 6, 9)$$

$$\theta_2 = \Theta(1, 2, 0) = (3, 5, 6)$$

$$\theta_3 = \Theta(2, 1, 0) = (2, 3, 6).$$

Then  $\underline{\theta} \in \text{Head}_{L_w,3}$ . We can find the structure of  $\mathbb{W}_{\underline{\theta}}$ , that is we can find the partitions  $\lambda_{\underline{\theta}}^{(1)}, \lambda_{\underline{\theta}}^{(2)}, \lambda_{\underline{\theta}}^{(3)}$  and  $\mu_{\underline{\theta}}^{(1)}, \mu_{\underline{\theta}}^{(2)}, \mu_{\underline{\theta}}^{(3)}$ . By summing the first entries from the alternate indexing of  $\theta_1, \theta_2$ , and  $\theta_3$  we see there will be 4 boxes in the skew semistandard tableaux  $T_{\underline{\theta}}$  that are in  $\text{Block}_{L_w,1}$  and so  $\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}$  will be a skew partition with 4 boxes. The first column of  $T_{\underline{\theta}}$  is associated with  $\theta_3 = \Theta(2, 1, 0)$  and so we see that the first two boxes in this column will be in  $\text{Block}_{L_w,1}$ . The second column of  $T_{\underline{\theta}}$  is associated with  $\theta_2 = \Theta(1, 2, 0)$  and so we see that the first box in this column will be in  $\text{Block}_{L_w,1}$ . The third column of  $T_{\underline{\theta}}$  is associated with  $\theta_1 = \Theta(1, 1, 1)$  and so we see that the first box in this column will be in  $\text{Block}_{L_w,1}$ . Thus  $\lambda_{\underline{\theta}}^{(1)} = (3, 1)$  and  $\mu_{\underline{\theta}}^{(1)} = (0)$ . The same can be done for the other partitions.

We can visualize this by marking the boxes of  $T_{\underline{\theta}}$ . We will mark the boxes associated with  $\text{Block}_{L_w,1}$  with a  $\star$ ,  $\text{Block}_{L_w,2}$  with a  $\bullet$ , and  $\text{Block}_{L_w,3}$  with a  $\dagger$ . Then, for example, the first column, associated to  $\theta_3 = \Theta(2, 1, 0)$  would have two boxes with a  $\star$  and one box with a  $\bullet$ . Thus we can visualize  $T_{\underline{\theta}}$  as

$\star$	$\star$	$\star$
$\star$	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\dagger$

From this it is trivial to see that

$$\mathbb{W}_{\underline{\theta}} = \mathbb{W}^{(3,1)} \otimes \mathbb{W}^{(3,2)/(1)} \otimes \mathbb{W}^{(1)}$$

**Theorem 4.0.17.** (Classification of multiplicity free Schubert varieties) Let  $w := (i_1, \dots, i_d) \in W^P$  such that  $i_1 \neq 1$  and  $i_d = N$ . Let  $X(w)$  be a Schubert variety in  $G_{d,N}$ .

- (a) If  $d'_{L_w} = 1$ , the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.
- (b) If  $d'_{L_w} = 2$ , the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.
- (c) If  $d'_{L_w} = 3$ , the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if  $h_1 + 1 = N_1$  or  $h_3 = 1$ .
- (d) If  $d'_{L_w} \geq 4$ , the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is not multiplicity free.

*Proof.* (a): When  $d'_{L_w} = 1$ ,  $X(w) = G_{d,N}$  and there is a single degree 1 head  $\underline{\theta}^1 = w \in \text{Head}_{L_w}$  with every entry in  $\text{Block}_{L_w,1}$ ; thus there is a single degree  $r$  head equal to  $\underline{\theta}^r := (\underline{\theta}^1, \dots, \underline{\theta}^1) \in \text{Head}_{L_w,r}$ . Thus criterion  $M2$  is trivially satisfied. Also, since there is only a single block we have  $\lambda_{\underline{\theta}^r}^{(1)}/\mu_{\underline{\theta}^r}^{(1)} = (r^d)/(0) = (r^d)$ . Because this is a partition  $\mathbb{W}_{\underline{\theta}^r} = \mathbb{W}^{\lambda_{\underline{\theta}^r}^{(1)}/\mu_{\underline{\theta}^r}^{(1)}} = \mathbb{W}^{(r^d)}$  is irreducible. Thus criterion  $M1$  is satisfied and this implies that  $\mathbb{C}[X(w)]$  is multiplicity free. In addition

$$\mathbb{C}[X(w)] = \bigoplus_{r \geq 1} \mathbb{C}[X(w)]_r = \bigoplus_{r \geq 1} \mathbb{W}_{\underline{\theta}^r}^* \otimes D_r \cong \bigoplus_{r \geq 1} \mathbb{W}^{((r^d)^*)^\pi} \cong \bigoplus_{r \geq 1} \mathbb{W}^{(r^{N-d})}.$$

(b): When  $d'_{L_w} = 2$  things become only slightly more complicated. Let  $\underline{\theta} \in \text{Head}_{L_w,r}$ . Then

$$\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}}.$$

Consider  $\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}$ . This skew diagram corresponds to those boxes in  $T_{\underline{\theta}}$  whose entries are in  $\text{Block}_{L_w,1}$ . If a row in  $T_{\underline{\theta}}$  has a box whose entry is in  $\text{Block}_{L_w,1}$  then the leftmost box in the row also must have its entry in  $\text{Block}_{L_w,1}$  since  $T_{\underline{\theta}}$  is a semistandard tableaux and in this case  $\text{Block}_{L_w,1} = \{1, \dots, k\}$  for some  $1 < k < N$ . This implies  $\mu_{\underline{\theta}}^{(1)} = (0)$ . Thus  $\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}}^{(1)}/(0) = \lambda_{\underline{\theta}}^{(1)}$ .

Next consider  $\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}$ . This skew diagram corresponds to those boxes in  $T_{\underline{\theta}}$  whose entries are in  $\text{Block}_{L_w,2}$ . If a row in  $T_{\underline{\theta}}$  has a box whose entry is in  $\text{Block}_{L_w,2}$  then the rightmost box in the row also must have its entry in  $\text{Block}_{L_w,2}$  since  $T_{\underline{\theta}}$  is a semistandard tableaux and in this case  $\text{Block}_{L_w,2} = \{k, \dots, N\}$  for some  $1 < k < N$ . Thus  $\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)} = (m^n)/\mu_{\underline{\theta}}^{(2)}$ , for  $m, n$  positive integers less than  $r, d$  respectively, and we have that

$$\mathbb{W}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}} = \mathbb{W}^{(m^n)/\mu_{\underline{\theta}}^{(2)}} \cong \mathbb{W}^{((m^n)/\mu_{\underline{\theta}}^{(2)})^\pi}$$

And thus

$$\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}^{((m^n)/\mu_{\underline{\theta}}^{(2)})^\pi}.$$

Also, since  $\lambda_{\underline{\theta}}^{(1)}$  and  $((m^n)/\mu_{\underline{\theta}}^{(2)})^\pi$  are both partitions this implies  $\mathbb{W}_{\underline{\theta}}$  is an irreducible  $L_w$ -module. Thus criterion  $M1$  is always satisfied.

Further, since  $\mathbb{W}_{\underline{\theta}}$  is an irreducible  $L_w$ -module for all  $\underline{\theta} \in \text{Head}_{L_w,r}$  we see that criterion  $M2$  may be simplified. We may check that for all  $r \geq 1$  and  $\underline{\theta}, \underline{\theta}' \in \text{Head}_{L_w,r}$ , if  $\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}_{\underline{\theta}'}$  then  $\underline{\theta} = \underline{\theta}'$ .

Thus fix an  $r \geq 1$  and let  $\underline{\theta}, \underline{\theta}' \in \text{Head}_{L_w,r}$ . Suppose that  $\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}_{\underline{\theta}'}$ . This implies that

$$\mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}^{((m^n)/\mu_{\underline{\theta}}^{(2)})^\pi} = \mathbb{W}^{\lambda_{\underline{\theta}'}^{(1)}} \otimes \mathbb{W}^{((p^q)/\mu_{\underline{\theta}'}^{(2)})^\pi}$$

which means, since these are all partitions, that  $\lambda_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}'}^{(1)}$  and  $((m^n)/\mu_{\underline{\theta}}^{(2)})^\pi = ((p^q)/\mu_{\underline{\theta}'}^{(2)})^\pi$ . The first of these two identities is enough for our purpose. Both  $T_{\underline{\theta}}$  and  $T_{\underline{\theta}'}$  are semistandard tableaux on the diagram  $(r^d)$ . So  $\lambda_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}'}^{(1)}$  implies that the boxes in  $T_{\underline{\theta}}$  whose entries are in  $\text{Block}_{L_w,1}$  are the same boxes in  $T_{\underline{\theta}'}$  whose entries are in  $\text{Block}_{L_w,1}$ . And since there are only two blocks it says the same about those boxes whose entries are in  $\text{Block}_{L_w,2}$ . The fact that  $\underline{\theta} = \underline{\theta}'$  is then a consequence of Lemma 3.1.8. And thus criterion  $M2$  is always satisfied and by Corollary 4.0.4 this implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.

(c): When  $d'_{L_w} = 3$  the decomposition once again becomes more complicated. Let  $\underline{\theta} \in \text{Head}_{L_w,r}$ . Then

$$\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}^{\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}} \otimes \mathbb{W}^{\lambda_{\underline{\theta}}^{(3)}/\mu_{\underline{\theta}}^{(3)}}$$

and by the same argument as in the  $d'_{L_w} = 2$  case we know that  $\lambda_{\underline{\theta}}^{(1)}/\mu_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}}^{(1)}$  and  $\lambda_{\underline{\theta}}^{(3)}/\mu_{\underline{\theta}}^{(3)} = (m^n)/\mu_{\underline{\theta}}^{(3)}$ , for  $m, n$  positive integers less than  $r, d$  respectively. Thus

$$\mathbb{W}_{\underline{\theta}} \cong \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}} \otimes \mathbb{W}_{\underline{\theta}}^{((m^n)/\mu_{\underline{\theta}}^{(3)})^\pi}.$$

**Criterion M2:** Fix an  $r \geq 1$  and let  $\underline{\theta}, \underline{\theta}' \in \text{Head}_{L_w, r}$ . Let  $I, I'$  be  $L_w$ -submodules of  $\mathbb{W}_{\underline{\theta}}, \mathbb{W}_{\underline{\theta}'}$  respectively. Then  $I \cong \mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}} \otimes \mathbb{W}_{\underline{\theta}}^{\nu_{\underline{\theta}}^{(2)}} \otimes \mathbb{W}_{\underline{\theta}}^{((m^n)/\mu_{\underline{\theta}}^{(3)})^\pi}$  where  $\mathbb{W}_{\underline{\theta}}^{\nu_{\underline{\theta}}^{(2)}}$  is an irreducible  $\text{GL}_{N_2}$ -submodule of  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}}$ . And  $I' \cong \mathbb{W}_{\underline{\theta}'}^{\lambda_{\underline{\theta}'}^{(1)}} \otimes \mathbb{W}_{\underline{\theta}'}^{\nu_{\underline{\theta}'}^{(2)}} \otimes \mathbb{W}_{\underline{\theta}'}^{((p^q)/\mu_{\underline{\theta}'}^{(3)})^\pi}$  where  $\mathbb{W}_{\underline{\theta}'}^{\nu_{\underline{\theta}'}^{(2)}}$  is an irreducible  $\text{GL}_{N_2}$ -submodule of  $\mathbb{W}_{\underline{\theta}'}^{\lambda_{\underline{\theta}'}^{(2)}/\mu_{\underline{\theta}'}^{(2)}}$ .

Now suppose that  $I \cong I'$ . This would imply that  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(1)}} \cong \mathbb{W}_{\underline{\theta}'}^{\lambda_{\underline{\theta}'}^{(1)}}$  which means that  $\lambda_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}'}^{(1)}$ .

Then  $I \cong I'$  also implies that  $\mathbb{W}_{\underline{\theta}}^{((m^n)/\mu_{\underline{\theta}}^{(3)})^\pi} \cong \mathbb{W}_{\underline{\theta}'}^{((p^q)/\mu_{\underline{\theta}'}^{(3)})^\pi}$  which in turn implies, since these are partitions, that  $((m^n)/\mu_{\underline{\theta}}^{(3)})^\pi = ((p^q)/\mu_{\underline{\theta}'}^{(3)})^\pi$ . Now by construction we know that both  $(m^n)/\mu_{\underline{\theta}}^{(3)}$  and  $(p^q)/\mu_{\underline{\theta}'}^{(3)}$  have no empty rows and columns. Thus by Lemma 2.3.5 we have that  $m = p$ ,  $n = q$ , and  $\mu_{\underline{\theta}}^{(3)} = \mu_{\underline{\theta}'}^{(3)}$ . As  $\lambda_{\underline{\theta}}^{(3)} = (m^n)$  and  $\lambda_{\underline{\theta}'}^{(3)} = (p^q)$  this implies that  $\lambda_{\underline{\theta}}^{(3)} = \lambda_{\underline{\theta}'}^{(3)}$ .

Both  $T_{\underline{\theta}}$  and  $T_{\underline{\theta}'}$  are semistandard tableaux on the diagram  $(r^d)$ . So the fact that  $\lambda_{\underline{\theta}}^{(1)} = \lambda_{\underline{\theta}'}^{(1)}$  implies that the boxes in  $T_{\underline{\theta}}$  whose entries are in  $\text{Block}_{L_w, 1}$  are the same boxes in  $T_{\underline{\theta}'}$  whose entries are in  $\text{Block}_{L_w, 1}$ . Then since  $\mu_{\underline{\theta}}^{(3)} = \mu_{\underline{\theta}'}^{(3)}$  and  $\lambda_{\underline{\theta}}^{(3)} = \lambda_{\underline{\theta}'}^{(3)}$  we have that the boxes in  $T_{\underline{\theta}}$  whose entries are in  $\text{Block}_{L_w, 3}$  are the same boxes in  $T_{\underline{\theta}'}$  whose entries are in  $\text{Block}_{L_w, 3}$ . As there are only three blocks these two results imply the same about those boxes with values in  $\text{Block}_{L_w, 2}$ . The fact that  $\underline{\theta} = \underline{\theta}'$  is then a consequence of Lemma 3.1.8. Thus criterion M2 is always satisfied when  $d'_{L_w} = 3$ .

**Criterion M1:** We now show that criterion M1 holds if and only if  $h_3 = 1$  or  $h_1 + 1 = N_1$ .

( $\Leftarrow$ ) Suppose that  $h_3 = 1$ . Then if  $w = (a_1, \dots, a_d)$  we know that  $a_j \notin \text{Block}_{L_w, 3}$  for  $j < d$ . But then for any head  $\theta = (i_1, \dots, i_d) \in \text{Head}_{L_w}$  we must have that  $i_j \notin \text{Block}_{L_w, 3}$  for  $j < d$ , since  $\theta < w$  implies that  $i_j \leq a_j$  for  $1 \leq j \leq d$ . This implies that for any  $r \geq 1$  and  $\underline{\theta} \in \text{Head}_{L_w, r}$ , the final row of  $T_{\underline{\theta}}$  is the only row that can contain entries from  $\text{Block}_{L_w, 3}$ . Thus we must have that  $\lambda_{\underline{\theta}}^{(2)} = (p^q, l)$  with  $l \leq p \leq r$  and  $q \leq d$ . Setting  $m = (\lambda_{\underline{\theta}}^{(2)})_1$  and  $n = (\lambda_{\underline{\theta}}^{(2)})'_1$  the  $m^n$ -complement,  $\lambda_{\underline{\theta}}^{(2)*} = (p - l)$ , is either a rectangle of  $m^n$ -shortness 1 or the zero partition. In either case, by Theorem 2.3.10 and Remark 2.4.3, this implies that  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}}$  is multiplicity free. Thus criterion M1 is satisfied.

Now suppose that  $h_1 + 1 = N_1$ . Any head  $\theta = (i_1, \dots, i_d) \in \text{Head}_{L_w}$  has  $i_j \in \text{Block}_{L_w, 1}$  for  $j \leq h_1$ ,  $i_{h_1+1}$  may be in  $\text{Block}_{L_w, 1}$ , and  $i_j \notin \text{Block}_{L_w, 1}$  for  $j > h_1 + 1$ . The last of these three is due to the fact that there are only  $N_1 = h_1 + 1$  possible distinct entries in  $\text{Block}_{L_w, 1}$  and  $\theta \in W^P$ . This implies that for any  $r \geq 1$  and  $\underline{\theta} \in \text{Head}_{L_w, r}$  the entries in the first  $h_1$  rows of  $T_{\underline{\theta}}$  are in  $\text{Block}_{L_w, 1}$ , and the only other boxes that can have entries in  $\text{Block}_{L_w, 1}$  are in row  $h_1 + 1$ . But this implies that  $\mu_{\underline{\theta}}^{(2)} = (p)$  for some  $0 \leq p \leq r$ . Setting  $m = (\lambda_{\underline{\theta}}^{(2)})_1$  and  $n = (\lambda_{\underline{\theta}}^{(2)})'_1$  we then have that  $\mu_{\underline{\theta}}^{(2)}$  is either a rectangle of  $m^n$ -shortness 1 or the zero partition. In either case, by Theorem 2.3.10 and Remark 2.4.3, this implies that  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}}$  is multiplicity free. Thus criterion M1 is satisfied.

Thus if either  $h_3 = 1$  or  $h_1 + 1 = N_1$  criterion M1 is satisfied. Further, as seen above, criterion M2 is always satisfied. Thus Corollary 4.0.4 implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.

( $\Rightarrow$ ) Suppose that  $h_3 \neq 1$  and  $h_1 + 1 \neq N_1$ . Then  $h_3 > 1$  and  $N_1 > h_1 + 1$ . We will show that criterion M1 is never satisfied. Let

$$\underline{\theta} := (\Theta(h_1, h_2, h_3), \Theta(h_1 + 1, h_2, h_3 - 1), \Theta(h_1 + 2, h_2, h_3 - 2)).$$

This is trivially verified to be a degree 3 head in  $\text{Head}_{L_w,3}$  (using the criterion from Lemma 4.0.14) since  $h_3 > 1$  and  $N_1 > h_1 + 1$ .

We are interested in finding the partitions  $\lambda_{\underline{\theta}}^{(2)}$  and  $\mu_{\underline{\theta}}^{(2)}$  by using reasoning similar to that of Example 4.0.16. We can visualize the Young tableau  $T_{\underline{\theta}}$  and the associated partitions, in the case where  $h_1 = 1$ ,  $h_2 = 1$ , and  $h_3 = 2$ , by marking the boxes. We will mark the boxes associated with  $\text{Block}_{L_w,1}$  with a  $\star$ ,  $\text{Block}_{L_w,2}$  with a  $\bullet$ , and  $\text{Block}_{L_w,3}$  with a  $\dagger$ .

$\star$	$\star$	$\star$
$\star$	$\star$	$\bullet$
$\star$	$\bullet$	$\dagger$
$\bullet$	$\dagger$	$\dagger$

In this case  $\lambda_{\underline{\theta}}^{(2)} = (3, 2, 1)$  and  $\mu_{\underline{\theta}}^{(2)} = (2, 1)$ . In general, we have that  $\lambda_{\underline{\theta}}^{(2)} = (3^{h_2}, 2, 1)$  and  $\mu_{\underline{\theta}}^{(2)} = (2, 1)$ . Setting  $m = (\lambda_{\underline{\theta}}^{(2)})_1 = 3$ , and  $n = (\lambda_{\underline{\theta}}^{(2)})'_1 = h_2 + 2$  we have that the  $m^n$ -complement  $(\lambda_{\underline{\theta}}^{(2)})^* = (2, 1)$ . As both  $\mu_{\underline{\theta}}^{(2)}$  and  $(\lambda_{\underline{\theta}}^{(2)})^*$  are not rectangles (or the zero partition), we know by Theorem 2.3.10 and Remark 2.4.3 that  $\mathbb{W}_{\underline{\theta}}^{\lambda_{\underline{\theta}}^{(2)}/\mu_{\underline{\theta}}^{(2)}}$  is not multiplicity free. Thus  $W_{\underline{\theta}}$  is not a multiplicity free  $L_w$ -module. We conclude then that  $M1$  is never satisfied in this case and thus by Corollary 4.0.4 this implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is not multiplicity free.

(d): Suppose  $d'_{L_w} \geq 4$ . We break this into two cases.

**Case 1:**  $d'_{L_w} = 4$ . We will show that criterion  $M2$  is never satisfied. Let

$$\underline{\theta} := (\Theta(h_1, h_2, h_3, h_4), \Theta(h_1, h_2 + 1, h_3, h_4 - 1), \Theta(h_1 + 1, h_2, h_3, h_4 - 1))$$

$$\underline{\theta}' := (\Theta(h_1, h_2, h_3, h_4), \Theta(h_1, h_2, h_3 + 1, h_4 - 1), \Theta(h_1 + 1, h_2 + 1, h_3 - 1, h_4 - 1)).$$

These are both trivially verified to be degree 3 heads in  $\text{Head}_{L_w,3}$  using the criterion from Lemma 4.0.14.

We are interested in finding the form of  $\mathbb{W}_{\underline{\theta}}$  using reasoning similar to that of Example 4.0.16. We can visualize the Young tableaux  $T_{\underline{\theta}}$  and  $T_{\underline{\theta}'}$  with their associated partitions, in the case where  $h_1 = 1$ ,  $h_2 = 1$ ,  $h_3 = 2$ , and  $h_4 = 2$ , by marking the boxes. We will mark the boxes associated with  $\text{Block}_{L_w,1}$  with a  $\star$ ,  $\text{Block}_{L_w,2}$  with a  $\bullet$ ,  $\text{Block}_{L_w,3}$  with a  $\dagger$ , and  $\text{Block}_{L_w,4}$  with a  $\diamond$ . The left figure is associated with  $T_{\underline{\theta}}$  and the right with  $T_{\underline{\theta}'}$ .

$\star$	$\star$	$\star$
$\star$	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\dagger$
$\dagger$	$\dagger$	$\dagger$
$\dagger$	$\dagger$	$\diamond$
$\diamond$	$\diamond$	$\diamond$

$\star$	$\star$	$\star$
$\star$	$\bullet$	$\bullet$
$\bullet$	$\dagger$	$\dagger$
$\bullet$	$\dagger$	$\dagger$
$\dagger$	$\dagger$	$\diamond$
$\diamond$	$\diamond$	$\diamond$

In this case we can see that

$$\mathbb{W}_{\underline{\theta}} = \mathbb{W}^{(3,1)} \otimes \mathbb{W}^{(3,2)/(1)} \otimes \mathbb{W}^{(3^2,2)/(2)} \otimes \mathbb{W}^{(3^2)/(2)}$$

and

$$\mathbb{W}_{\underline{\theta}'} = \mathbb{W}^{(3,1)} \otimes \mathbb{W}^{(3,1,1)/(1)} \otimes \mathbb{W}^{(3^2,2)/(1,1)} \otimes \mathbb{W}^{(3^2)/(2)}.$$

In general when  $h_3 > 1$  and  $h_4 > 1$  we have

$$\mathbb{W}_{\underline{\theta}} = \mathbb{W}^{(3^{h_1},1)} \otimes \mathbb{W}^{(3^{h_2},2)/(1)} \otimes \mathbb{W}^{(3^{h_3},2)/(2)} \otimes \mathbb{W}^{(3^{h_4})/(2)}$$

and



$$\mathbb{W}_{\underline{\theta}'} = \mathbb{W}^{(3^{h_1}, 1)} \otimes \mathbb{W}^{(3^{h_2}, 1, 1)/(1)} \otimes \mathbb{W}^{(3^{h_3}, 2)/(1, 1)} \otimes \mathbb{W}^{(3^{h_4})/(2)}.$$

These  $L_w$ -modules are the same except in the second and third tensor terms. But then  $\mathbb{W}^{(3^{h_2}, 1)}$  is a  $\mathrm{GL}_{N_2}$ -submodule of  $\mathbb{W}^{(3^{h_2}, 2)/(1)}$  and of  $\mathbb{W}^{(3^{h_2}, 1, 1)/(1)}$ . In addition,  $\mathbb{W}^{(3^{h_3-1}, 2, 1)}$  is a  $\mathrm{GL}_{N_3}$ -submodule of  $\mathbb{W}^{(3^{h_3}, 2)/(2)}$  and of  $\mathbb{W}^{(3^{h_3}, 2)/(1, 1)}$ . This may be verified by checking that the Littlewood-Richardson coefficients associated to the triples are always 1 (cf. Lemma 2.5.3). But this implies that

$$\mathbb{W}^{(3^{h_1}, 1)} \otimes \mathbb{W}^{(3^{h_2}, 1)} \otimes \mathbb{W}^{(3^{h_3-1}, 2, 1)} \otimes \mathbb{W}^{(3^{h_4})/(2)}$$

is a  $L_w$ -submodule of both  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\underline{\theta}'}$ . And thus criterion  $M2$  is not satisfied.

When  $h_4 = 1$  the module structures of  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\underline{\theta}'}$  are exactly the same except the skew diagram  $(3^{h_4})/(2) = (3)/(2)$  has empty columns. So in this case when the empty columns are deleted the fourth tensor term in both  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\underline{\theta}'}$  equals  $\mathbb{W}^{(1)}$ .

When  $h_3 = 1$  the module structures of  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\underline{\theta}'}$  are exactly the same except that the skew diagram  $(3^{h_3}, 2)/(1, 1) = (3, 2)/(1, 1)$  has an empty column. So in this case when the empty column is deleted the third tensor term in  $\mathbb{W}_{\underline{\theta}'}$  equals  $\mathbb{W}^{(2, 1)}$ . We have  $\mathbb{W}^{(2, 1)}$  is a  $\mathrm{GL}_{N_3}$ -submodule of  $\mathbb{W}^{(3, 2)/(2)}$  and equals  $\mathbb{W}^{(2, 1)}$ . This may be verified by checking that the Littlewood-Richardson coefficients associated to the triples are always 1 (cf. Lemma 2.5.3).

This implies that

$$\mathbb{W}^{(3^{h_1}, 1)} \otimes \mathbb{W}^{(3^{h_2}, 1)} \otimes \mathbb{W}^{(2, 1)} \otimes \mathbb{W}^{(3^{h_4})/(2)}$$

is a  $L_w$ -submodule of both  $\mathbb{W}_{\underline{\theta}}$  and  $\mathbb{W}_{\underline{\theta}'}$ .

And thus criterion  $M2$  is not satisfied and by Corollary 4.0.4 this implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is not multiplicity free.

**Case 2:**  $d'_{L_w} > 4$ . We will show that criterion  $M2$  is never satisfied. Let

$$\underline{\theta} := (\Theta(h_1, h_2, h_3, h_4, h_5, \dots, h_{d'_{L_w}}), \Theta(h_1, h_2 + 1, h_3, h_4 - 1, h_5, \dots, h_{d'_{L_w}}), \Theta(h_1 + 1, h_2, h_3, h_4 - 1, h_5, \dots, h_{d'_{L_w}}))$$

$$\underline{\theta}' := (\Theta(h_1, h_2, h_3, h_4, h_5, \dots, h_{d'_{L_w}}), \Theta(h_1, h_2, h_3 + 1, h_4 - 1, h_5, \dots, h_{d'_{L_w}}), \Theta(h_1 + 1, h_2 + 1, h_3 - 1, h_4 - 1, h_5, \dots, h_{d'_{L_w}}))$$

These are both trivially verified to be degree 3 heads using the criterion from Lemma 4.0.14.

Note that entries of these degree 3 heads which are in the first four blocks are exactly the same as those from the previous case. And those entries in the fifth block onwards are the same for  $\underline{\theta}$  and  $\underline{\theta}'$ . But this means we will get a violation of criterion  $M2$  in exactly the same way as in the previous case. Thus by Corollary 4.0.4 this implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is not multiplicity free.  $\square$

**Example 4.0.18.** Consider the Schubert variety  $X(w)$  in  $G_{3,9}$  where  $w = (3, 6, 9)$ . Then  $R'_{Q_w} = \{3, 6\}$  and so  $d'_{L_w} = 3$ . Further,  $h_1 = h_2 = h_3 = 1$ . Since the first entry of  $w$  is not 1 and the final entry is equal to 9 we may apply Theorem 4.0.17 (c) to conclude that  $\mathbb{C}[X(w)]$  has a decomposition into irreducible  $L_w$ -modules that is multiplicity free.

Alternatively, consider the Schubert variety  $X(w)$  in  $G_{7,15}$  where  $w = (1, 2, 5, 6, 9, 12, 14)$ . Since the first entry of  $w$  is 1 we apply Proposition 4.0.5(b) to see that  $\mathbb{C}[X(w)]$  has a decomposition into irreducible  $L_w$ -modules that is multiplicity free if and only if  $\mathbb{C}[X(\overline{w})]$  has a decomposition into irreducible  $L_{\overline{w}}$ -modules that is multiplicity free, where  $\overline{w} = (3, 4, 7, 10, 12)$  and  $X(\overline{w})$  is a Schubert variety in  $G_{5,12}$ . Then  $R'_{Q_{\overline{w}}} = \{4, 7, 10\}$  and so  $d'_{L_{\overline{w}}} = 4$ . Thus we conclude by Theorem 4.0.17 (d) that  $\mathbb{C}[X(\overline{w})]$  does not have a decomposition into irreducible  $L_{\overline{w}}$ -modules that is multiplicity free.

**Remark 4.0.19.** Given a Schubert variety  $X(w)$  in  $G_{d,N}$ , say  $w = (i_1, \dots, i_d)$ , we associate a partition  $\Lambda_w := (\Lambda_1, \dots, \Lambda_d)$ ,  $\Lambda_r = i_{d+1-r} - (d+1-r)$ . We shall denote  $X(w)$  also by  $X(\Lambda_w)$ . Under the canonical isomorphism  $G_{d,N} \cong G_{N-d,N}$ , induced by Weyl involution  $-w_0$ ,  $w_0$  being the largest element in  $S_N$ ,  $X(\Lambda_w)$  is mapped isomorphically onto  $X(\Lambda'_w)$ ,  $\Lambda'_w$  being the partition conjugate to  $\Lambda_w$ . This is the source of the two conditions  $h_1 + 1 = N, h_3 = 1$  in (c) of the Theorem 4.0.17, as  $h_1 + 1 = N$  for  $X(\Lambda_w)$  forces  $h_3 = 1$  for  $X(\Lambda'_w)$ , and vice versa.

We are able to derive a number of important corollaries from the classification theorem.

**Corollary 4.0.20.** *Let  $X(w)$  be a Schubert variety in  $G_{d,N}$ . If  $X(w)$  is smooth, the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* If  $X(w)$  is smooth then  $w$  is of the form

$$(1, \dots, p, m+1, \dots, m+i)$$

for  $0 \leq p, i \leq d$  and  $m < N$  such that  $m+i \leq N$  and  $p+i = d$ . Set  $\bar{w} = (m-p+1, \dots, m+i-p)$ . When  $p = 0$  the sequence  $1, \dots, p$  is taken to be empty. We have  $X(\bar{w}) = G_{\bar{d}, \bar{N}}$  for  $\bar{d} = i$  and  $\bar{N} = m+i-p$ . By Proposition 4.0.5(b) the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free, where  $L_{\bar{w}}$  is the Levi part of the stabilizer  $Q_{\bar{w}}$  of  $X(\bar{w})$ . Let  $\bar{d}'_{L_{\bar{w}}}$  be the number of blocks of  $L_{\bar{w}}$ , then  $\bar{d}'_{L_{\bar{w}}} = |R'_{Q_{\bar{w}}}| + 1 = 1$ . We conclude by Theorem 4.0.17(a) that the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free.  $\square$

Let  $B^-$  be the subgroup of lower triangular matrices in  $GL_N$ . Then  $B^-[e_{id}]$  is a dense open subset of  $G_{d,N}$ , called *the opposite cell* in  $G_{d,N}$ . For a Schubert variety  $X(w)$  in  $G_{d,N}$ , let  $Y(w) := B^-[e_{id}] \cap X(w)$ ; note that  $Y(w)$  is non-empty, since  $[e_{id}] \in B^-[e_{id}] \cap X(w)$ . We have that  $Y(w)$  is an open affine subvariety of  $X(w)$  and is usually called *the opposite cell* in  $X(w)$ . We have that

$$B^-[e_{id}] = \left\{ \begin{bmatrix} Id_{d \times d} & 0_{d \times N-d} \\ X_{N-d \times d} & Id_{N-d \times N-d} \end{bmatrix} \in GL_N \right\}$$

$X_{N-d \times d}$  being a generic  $N-d \times d$  matrix. Thus we obtain a natural identification of  $M_{N-d,d}(\mathbb{C})$ , the space of  $N-d \times d$  matrices (over  $\mathbb{C}$ ), with the dense open subset  $B^-[e_{id}]$  of  $G_{d,N}$ .

**Definition 4.0.21.** Let  $1 \leq k < \min(d, N-d)$ . The *determinantal variety*  $D_k(\mathbb{C})$  is the subset of  $M_{N-d,d}(\mathbb{C})$ , consisting of all  $N-d \times d$  matrices over  $\mathbb{C}$  with rank  $\leq k$ . We have that under the above identification of  $M_{N-d,d}(\mathbb{C})$  with  $B^-[e_{id}]$ ,  $D_k(\mathbb{C})$  gets identified with  $Y(w)$  for  $w = (k+1, \dots, d, N-k+1, \dots, N)$  (cf. [Ses14, Section 1.6]). The Schubert varieties with  $w = (k+1, \dots, d, N-k+1, \dots, N)$  are referred to as *determinantal Schubert varieties*. Note that these are precisely the Schubert varieties which are  $P_{\hat{\alpha}}$ -stable for left multiplication.

**Corollary 4.0.22.** *Let  $X(w)$  be a determinantal Schubert variety in  $G_{d,N}$ . Then the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* Since  $w$  is of the form  $(k+1, \dots, d, N-k+1, \dots, N)$  for  $1 \leq k < \min(d, N-d)$  the first entry in  $w$  does not equal 1 and the final entry equals  $N$ . Thus we may use Theorem 4.0.17. In this case  $d'_{L_w} = |R'_{Q_w}| + 1 = |\{d\}| + 1 = 2$  and we conclude that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.  $\square$

The above corollary relies on the fact that for the determinantal Schubert varieties  $d' = 2$ . The converse is not always true, that is the Schubert varieties where  $d'_{L_w} = 2$  are not all determinantal. For example  $X((3, 4, 6)) \subset G_{3,6}$  has  $d'_{L_w} = 2$  but  $X((3, 4, 6))$  is not a determinantal Schubert variety.

**Corollary 4.0.23.** *Let  $X(w)$  be a Schubert variety in  $G_{2,N}$ . Then the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* Define  $\bar{w}$  as in Proposition 4.0.5. Then by Proposition 4.0.5(b) the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free, where  $L_{\bar{w}}$  is the Levi part of the stabilizer  $Q_{\bar{w}}$  of  $X(\bar{w})$ .

The Schubert variety  $X(\bar{w}) \subset G_{\bar{d},\bar{N}}$  for  $\bar{d} \leq d$  and  $\bar{N} \leq N$ . Let  $\bar{d}'_{L_{\bar{w}}}$  be the number of blocks of  $L_{\bar{w}}$ , then by Lemma 4.0.9  $\bar{d}'_{L_{\bar{w}}} \leq \bar{d} \leq d = 2$ . We conclude by applying Theorem 4.0.17(a)(b) to see that the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free.  $\square$

**Corollary 4.0.24.** *Let  $X(w)$  be a Schubert variety in  $G_{3,N}$ . Then the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* Define  $\bar{w}$  as in Proposition 4.0.5. Then by Proposition 4.0.5(b) the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free if and only if the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free, where  $L_{\bar{w}}$  is the Levi part of the stabilizer  $Q_{\bar{w}}$  of  $X(\bar{w})$ .

We have  $X(\bar{w}) \subset G_{\bar{d},\bar{N}}$  for  $\bar{d} \leq d$  and  $\bar{N} \leq N$ . Let  $\bar{d}'_{L_{\bar{w}}}$  be the number of blocks of  $L_{\bar{w}}$ , then by Lemma 4.0.9  $\bar{d}'_{L_{\bar{w}}} \leq \bar{d} \leq d = 3$ . If  $\bar{d}'_{L_{\bar{w}}} \leq 2$  we conclude by applying Theorem 4.0.17(a)(b) to see that the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free. If  $\bar{d}'_{L_{\bar{w}}} = 3$  then we know that the number of entries of  $\bar{w}$  in the third block,  $h_3$ , equals 1. Thus by Theorem 4.0.17(c) the decomposition of  $\mathbb{C}[X(\bar{w})]$  into irreducible  $L_{\bar{w}}$ -modules is multiplicity free.  $\square$

## 5. SPHERICITY CONSEQUENCES OF THE DECOMPOSITION

For this section fix  $d, N$  positive integers with  $d < N$ . Let  $P = P_{\hat{d}}$  and  $w = (i_1, \dots, i_d) \in W^P$ . Let  $L_w$  be the Levi part of the stabilizer  $Q_w$  of  $X(w)$ .

The multiplicity results from the previous section can be recast into results about the sphericity of Schubert varieties. Let  $G$  be a reductive group with  $B$  a Borel subgroup. Suppose that  $X$  is an irreducible  $G$ -variety. Then  $X$  is a *spherical  $G$ -variety* if it is normal and it has an open dense  $B$ -orbit(cf. [BLV86]).

We wish to relate the sphericity of a projective variety  $X \hookrightarrow \mathbb{P}(V)$  and the cone  $\hat{X}$  over  $X$ .

**Proposition 5.0.1.** *Let  $X$  be projectively normal, namely  $\mathbb{C}[X]$ , the homogeneous coordinate ring of  $X$ , is normal. Let  $G$  be a reductive group acting linearly on  $X$ , that is  $V$  is a  $G$ -module and the action of  $G$  on  $X$  is induced from the  $G$ -action on  $\mathbb{P}(V)$ . If  $\mathbb{C}[X]$  has a multiplicity free decomposition into irreducible  $G$ -modules, then  $\hat{X}, X$  are spherical  $G$ -varieties.*

*Proof.* We have, by [Tim11, Theorem 25.1], that an affine normal  $G$ -variety ( $G$  reductive) is a spherical  $G$ -variety if and only if the decomposition of its coordinate ring into irreducible  $G$ -modules is multiplicity free. Thus  $\hat{X}$  is a spherical  $G$ -variety and hence has a dense open  $B$ -orbit  $U$ .

The canonical map  $\pi : V \setminus \{0\} \rightarrow \mathbb{P}(V)$  is  $G$ -equivariant. Denoting the restriction of  $\pi$  to  $\hat{X} \setminus \{0\}$  by  $\pi'$ , we get a  $G$ -equivariant map  $\pi' : \hat{X} \setminus \{0\} \rightarrow X$ . The map  $\pi'$  is a principal fiber bundle for the action of the multiplicative group  $\mathbb{G}_m$ , and therefore a geometric quotient. Hence  $\pi'$  is an open map and we get that  $U' := \pi'(U)$  is open and dense in  $X$ . As  $U'$  is the image of a  $B$ -orbit under a  $G$ -equivariant map we have that  $U'$  is itself a  $B$ -orbit. Thus  $X$  is a spherical  $G$ -variety.  $\square$

In light of this result we may interpret our classification theorem found in the previous section in terms of the sphericity of  $L_w$ -varieties. As in the previous section we restrict our results to the case where our group  $L$  is the Levi part of the full stabilizer  $Q_w$  of  $X(w)$ .

**Corollary 5.0.2.** *Let  $X(w)$  be a Schubert variety in  $G_{d,N}$ . If  $X(w)$  is smooth, then  $X(w)$  is a spherical  $L_w$ -variety.*

*Proof.* By Corollary 4.0.20 we have that  $X(w)$  smooth implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free. By Proposition 5.0.1 this implies  $X(w)$  is a spherical  $L_w$ -variety.  $\square$

**Corollary 5.0.3.** *Let  $X(w)$  be a determinantal Schubert variety in  $G_{d,N}$ . Then  $X(w)$  is a spherical  $L_w$ -variety.*

*Proof.* By Corollary 4.0.22 we have that  $X(w)$  a determinantal Schubert variety implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free. By Proposition 5.0.1 this implies  $X(w)$  is a spherical  $L_w$ -variety.  $\square$

**Corollary 5.0.4.** *Let  $1 \leq k < \min(d, N-d)$  and  $w = (k+1, \dots, d, N-k+1, \dots, N)$ . Then the determinantal variety  $D_k(\mathbb{C})$  is  $L_w$ -stable and is a spherical  $L_w$ -variety.*

*Proof.* As in Definition 4.0.21,  $D_k(\mathbb{C})$  is realized as  $Y(w)$  for  $w = (k+1, \dots, d, N-k+1, \dots, N)$ . By Corollary 5.0.3 we have that the determinantal Schubert variety  $X(w)$  is a spherical  $L_w$ -variety. Also, as noted in Definition 4.0.21, we have  $Q_w = P_{\hat{d}}$  and  $L_w = \mathrm{GL}_d \times \mathrm{GL}_{N-d}$ . Further,  $B^-[e_{id}]$  is  $\mathrm{GL}_d \times \mathrm{GL}_{N-d}$ -stable. Hence  $D_k(\mathbb{C})$  is a  $L_w$ -stable sub variety of  $X(w)$ . Thus  $Y(w)$  is an open  $L_w$ -stable subvariety of  $X(w)$ , and hence is a spherical  $L_w$ -variety.  $\square$

**Remark 5.0.5.** As a subvariety of  $M_{N-d,d}(\mathbb{C})$ , the action of  $L_w (= \mathrm{GL}_d \times \mathrm{GL}_{N-d})$  on  $D_k(\mathbb{C})$  is induced from the natural action of  $\mathrm{GL}_d \times \mathrm{GL}_{N-d}$  on  $M_{N-d,d}(\mathbb{C})$ :  $(A, B) \cdot X = BXA^{-1}$ ,  $A \in \mathrm{GL}_d$ ,  $B \in \mathrm{GL}_{N-d}$ ,  $X \in M_{N-d,d}(\mathbb{C})$ .

**Corollary 5.0.6.** *Let  $1 \leq k < \min(d, N-d)$  and  $w = (k+1, \dots, d, N-k+1, \dots, N)$ . Then the decomposition of  $\mathbb{C}[D_k(\mathbb{C})]$  into irreducible  $L_w$ -modules is multiplicity free.*

*Proof.* By Corollary 5.0.4,  $D_k(\mathbb{C})$  is a spherical  $L_w$ -variety. Further,  $D_k(\mathbb{C})$  being an affine variety, the result follows from [Tim11, Theorem 25.1].  $\square$

**Corollary 5.0.7.** *Let  $X(w)$  be a Schubert variety in  $G_{d,N}$  for  $d = 2, 3$ . Then  $X(w)$  is a spherical  $L_w$ -variety.*

*Proof.* By Corollary 4.0.23 and Corollary 4.0.24 we have that  $X(w)$  a Schubert variety in  $G_{d,N}$  for  $d = 2, 3$  implies that the decomposition of  $\mathbb{C}[X(w)]$  into irreducible  $L_w$ -modules is multiplicity free. By Proposition 5.0.1 this implies  $X(w)$  is a spherical  $L_w$ -variety.  $\square$

## 6. SINGULARITIES AND THE L-ACTION IN DEGREE 1

**6.1. The Smooth and Singular Locus.** There is an interesting relationship between the degree 1 heads of type  $L_w$  and the singularities of  $X(w)$ . To make the connection concrete we first need the following definition and lemma.

**Definition 6.1.1.** Let  $\theta \in \mathrm{Head}_{L_w}$ , then  $\theta$  is a *maximal degree 1 head* if there exists no  $\theta' \in \mathrm{Head}_{L_w}$  such that  $\theta < \theta' < w$ .

**Lemma 6.1.2.** *The head  $\theta \in \mathrm{Head}_{L_w}$  is a maximal degree 1 head if and only if*

$$\theta = \Theta(h_1, \dots, h_{i-1}, h_i + 1, h_{i+1} - 1, h_{i+2}, \dots, h_{d'_{L_w}})$$

*for some  $1 \leq i < d'_{L_w}$ , where the  $h_k$  are defined as in (4.0.10).*

*Proof.* ( $\Leftarrow$ ) Suppose

$$\theta = \Theta(n_1, \dots, n_{d'_{L_w}}) = \Theta(h_1, \dots, h_{i-1}, h_i + 1, h_{i+1} - 1, h_{i+2}, \dots, h_{d'_{L_w}})$$

for some  $1 \leq i < d'_{L_w}$ . Then clearly the nonnegative integers  $n_1, \dots, n_{d'_{L_w}}$  satisfy the conditions of Lemma 4.0.14, so  $\theta \in \text{Head}_{L_w}$ . Now suppose that  $\theta' = \Theta(m_1, \dots, m_{d'_{L_w}})$  is another degree 1 head such that  $\theta < \theta' < w$ . This implies that  $n_1 + \dots + n_k \geq m_1 + \dots + m_k \geq h_1 + \dots + h_k$  for all  $1 \leq k \leq d'_{L_w}$ . But this implies that  $h_1 = n_1 \geq m_1 \geq h_1$ , that is  $m_1 = h_1$ . It follows, by induction, that  $m_j = h_j$  for  $j < i$ . This implies that  $h_i + 1 = n_i \geq m_i \geq h_i$ , and so we have the following two possible cases.

**Case 1:**  $m_i = h_i$ . In this case we have

$$\begin{aligned} n_1 + \dots + n_{i+1} &\geq m_1 + \dots + m_{i+1} \geq h_1 + \dots + h_{i+1} \\ h_i + 1 + h_{i+1} - 1 &\geq h_i + m_{i+1} \geq h_i + h_{i+1} \\ h_{i+1} &\geq m_{i+1} \geq h_{i+1} \end{aligned}$$

Thus  $m_{i+1} = h_{i+1}$ . Continuing in this way we see that in this case  $m_j = h_j$  for  $j \geq i$ , and thus  $\Theta(m_1, \dots, m_{d'_{L_w}}) = \Theta(h_1, \dots, h_{d'_{L_w}})$ . This contradicts our choice of  $\theta' < w$ .

**Case 2:**  $m_i = h_i + 1$ . In this case we have

$$\begin{aligned} n_1 + \dots + n_{i+1} &\geq m_1 + \dots + m_{i+1} \geq h_1 + \dots + h_{i+1} \\ h_i + 1 + h_{i+1} - 1 &\geq h_i + 1 + m_{i+1} \geq h_i + h_{i+1} \\ h_{i+1} - 1 &\geq m_{i+1} \geq h_{i+1} - 1 \end{aligned}$$

Thus  $m_{i+1} = h_{i+1} - 1$ . Continuing in this way, we see that in this case  $m_j = h_j$  for  $j > i + 1$ , and thus  $\Theta(m_1, \dots, m_{d'_{L_w}}) = \Theta(n_1, \dots, n_{d'_{L_w}})$ . This contradicts our choice of  $\theta < \theta'$ .

In either case we arrive at a contradiction, and thus no choice of a head  $\theta'$  exists such that  $\theta < \theta' < w$ , so  $\theta$  is a maximal degree 1 head.

( $\Rightarrow$ ) Suppose that  $\theta \in \text{Head}_{L_w}$  is a maximal degree 1 head. We have that  $\theta = \Theta(m_1, \dots, m_{d'_{L_w}})$  for some nonnegative integers  $m_1, \dots, m_{d'_{L_w}}$  satisfying the conditions of Lemma 4.0.14. Thus we have that  $m_1 + \dots + m_k \geq h_1 + \dots + h_k$  for all  $1 \leq k \leq d'_{L_w}$ . Let  $j$  be the minimal index such that  $m_1 + \dots + m_j > h_1 + \dots + h_j$ . Then  $m_j = h_j + n$  for some  $n \geq 1$ . Consider the head  $\Theta(n_1, \dots, n_{d'_{L_w}}) = \Theta(h_1, \dots, h_{j-1}, h_j + 1, h_{j+1} - 1, h_{j+2}, \dots, h_{d'_{L_w}})$ . Then

$$m_1 + \dots + m_k = n_1 + \dots + n_k \text{ for } 1 \leq k < j$$

and

$$m_1 + \dots + m_j \geq n_1 + \dots + n_j$$

with

$$m_1 + \dots + m_k \geq h_1 + \dots + h_k = n_1 + \dots + n_k \text{ for } j < k \leq d'_{L_w}.$$

Thus  $\Theta(m_1, \dots, m_{d'_{L_w}}) \leq \Theta(n_1, \dots, n_{d'_{L_w}})$ . As  $\Theta(m_1, \dots, m_{d'_{L_w}})$  is a maximal degree 1 head this must in fact be an equality.  $\square$

Note that a trivial consequence of this lemma is that there are precisely  $d'_{L_w} - 1$  maximal degree 1 heads.

A point  $x$  on a variety  $X$  is said to be a smooth point of  $X$  if  $\mathcal{O}_{X,x}$  is a regular local ring, that is, the maximal ideal has a set of  $n = \dim \mathcal{O}_{X,x}$  generators. A point  $x$  that is not smooth is called a singular point of  $X$ . The singular locus and smooth locus of  $X$  are defined as

$$\begin{aligned}\text{Sing}X &:= \{x \in X \mid x \text{ is a singular point of } X\} \\ \text{Sm}X &:= \{x \in X \mid x \text{ is a smooth point of } X\}.\end{aligned}$$

A convenient method for describing  $\text{Sing}X(w)$  can be found in [LW90]. We may write the partition  $\Lambda_w$ , defined in Remark 4.0.19, as

$$\Lambda_w = (p_1^{q_1}, \dots, p_r^{q_r}) = (\underbrace{p_1, \dots, p_1}_{q_1}, \dots, \underbrace{p_r, \dots, p_r}_{q_r}).$$

**Theorem 6.1.3.** (Theorem 5.3 of [LW90]) *With the above notation,  $\text{Sing}X(w)$  has  $r-1$  components  $X(\alpha_1), \dots, X(\alpha_{r-1})$ , where the  $\alpha_j$  are partitions of the form*

$$\alpha_j = (p_1^{q_1}, \dots, p_{j-1}^{q_{j-1}}, p_j^{q_j-1}, (p_{j+1} - 1)^{q_{j+1}+1}, p_{j+1}^{q_{j+1}}, \dots, p_r^{q_r}).$$

**Proposition 6.1.4.** *With the above notation,  $\alpha_j = \Lambda_{\tau_j}$ , where*

$$\tau_j = \Theta(h_1, \dots, h_{j-1}, h_j + 1, h_{j+1} - 1, h_{j+2}, \dots, h_{d'_{L_w}}).$$

*Further,  $d'_{L_w} = r$  and the maximal degree 1 heads index the singular components of  $X(w)$ , that is,  $X(\tau_j)$  for  $1 \leq j < d'_{L_w}$  are the singular components of  $X(w)$ .*

*Proof.* We have  $w = \Theta(h_1, \dots, h_{d'_{L_w}})$ . Define the variables

$$M_i = N_1 + \dots + N_i - (h_1 + \dots + h_i) \text{ for } 1 \leq i \leq d'_{L_w}.$$

Then

$$\Lambda_w = ((M_{d'_{L_w}})^{h_{d'_{L_w}}}, \dots, (M_1)^{h_1}).$$

Thus  $d'_{L_w} = r$ . The maximal heads have the form

$$\tau_j = \Theta(h_1, \dots, h_{j-1}, h_j + 1, h_{j+1} - 1, h_{j+2}, \dots, h_{d'_{L_w}}).$$

for some  $1 \leq j < d'_{L_w}$  with

$$\Lambda_{\tau_j} = ((M_{d'_{L_w}})^{h_{d'_{L_w}}}, \dots, (M_{j+2})^{h_{j+2}}, (M_{j+1})^{h_{j+1}-1}, (M_j - 1)^{h_j+1}, (M_{j-1})^{h_{j-1}}, \dots, (M_1)^{h_1})$$

which precisely matches the partition  $\alpha_j$  associated to the singular component  $X(\alpha_j)$  of  $X(w)$ .  $\square$

**Corollary 6.1.5.**

(a) *The singular locus*

$$\text{Sing}X(w) = \bigcup_{\substack{\theta \in \text{Head}_{L_w} \\ \theta \text{ a maximal degree 1 head}}} X(\theta)$$

(b) *The singular locus of  $X(w)$  is  $L_w$ -stable.*

(c) *The smooth locus of  $X(w)$  is  $L_w$ -stable.*

**6.2. Multiplicity at a Point.** Below we recall the definition of the multiplicity of a variety at a point, as well as a few important results about multiplicities. For a more in depth introduction see [BL00]. Let  $K$  be an algebraically closed field and  $A$  a local, finitely generated  $K$ -algebra, with unique maximal ideal  $\mathfrak{m}$ . The Hilbert-Samuel function of  $A$  is defined to be

$$F_A(n) = \text{length}(A/\mathfrak{m}^n) (= \dim_K(A/\mathfrak{m}^n)).$$

**Theorem 6.2.1.** *There exists a polynomial  $P_A(x) \in \mathbb{Q}[x]$  of degree  $n = \dim A$ , called the Hilbert-Samuel polynomial of  $A$ , such that  $P_A(l) = F_A(l)$  for  $l \gg 0$ . Further, the leading coefficient of  $P_A(x)$  is of the form  $m_A/n!$ , where  $m_A$  is a positive integer.*

**Definition 6.2.2.** The number  $m_A$ , defined in Theorem 6.2.1, is called the multiplicity of  $A$ .



**Definition 6.2.3.** The multiplicity of an algebraic variety  $X$  at the point  $P$  is defined to be  $m_{\mathcal{O}_{X,P}}$ , the multiplicity of the stalk at  $P$ ,  $\mathcal{O}_{X,P}$ , and is denoted  $\text{mult}_P(X)$ .

**Proposition 6.2.4.** ([Mum76]) *Let  $P$  be a point on an algebraic variety  $X$ . Then  $P$  is a smooth point of  $X$  if and only if  $\text{mult}_P(X(w)) = 1$ .*

In this section we will refer to the  $T$ -fixed point  $[e_\tau] \in X(w)$  by  $e_\tau$  to simplify the notation. Given a  $P \in X(w)$ , let  $e_\tau$  be the  $T$ -fixed point of the  $B$ -orbit through  $P$ . We shall denote  $\text{mult}_{e_\tau}(X(w))$  as  $\text{mult}_\tau(w)$ . We have that  $\text{mult}_P(X(w)) = \text{mult}_\tau(w)$ . Thus it is sufficient to compute  $\text{mult}_\tau(X(w))$  for all  $T$ -fixed points in  $X(w)$ . This is done in [LW90] for all Schubert varieties in  $G/P$ , for  $P$  a maximal parabolic subgroup of minuscule type.

In our situation, the  $L_w$ -action gives some insight into which  $T$ -fixed points have equal multiplicities. Fix  $d, N$  positive integers with  $d < N$ . Let  $P = P_{\hat{d}}$  and  $w = (i_1, \dots, i_d) \in W^P$ . Let  $L_w$  be the Levi part of the stabilizer  $Q_w$  of  $X(w)$ .

**Proposition 6.2.5.** *Let  $\tau \in H_w$ . Then  $\theta_\tau = w$  if and only if  $\text{mult}_\tau(w) = 1$ .*

*Proof.* ( $\Rightarrow$ ) First observe that  $\tau \not\leq \xi$  for all  $\xi \in \text{Head}_{L_w}$  such that  $\xi \neq w$ . For, if  $\tau \leq \xi$  for some  $\xi \in \text{Head}_{L_w}$  with  $\xi \neq w$ , then by Proposition 3.2.2 we have  $w = \theta_\tau \leq \theta_\xi = \xi$ , which contradicts our choice of  $\xi$ . By Proposition 6.1.4 this implies  $e_\tau \notin \text{Sing}X(w)$ , which by Proposition 6.2.4 implies that  $\text{mult}_\tau(w) = 1$ .

( $\Leftarrow$ ) If  $\text{mult}_\tau(w) = 1$ , then by Proposition 6.2.4 we have  $e_\tau \notin \text{Sing}X(w)$ . Thus by Corollary 6.1.5(a) we have that  $\tau \not\leq \xi$  for all  $\xi \in \text{Head}_{L_w}$  such that  $\xi$  is a maximal degree 1 head. But this means that  $\theta_\tau = w$ .  $\square$

**Corollary 6.2.6.**

- (a) *The set of  $T$ -fixed points in the smooth locus of  $X(w)$  is equal to  $E_w^{L_w} := \{e_\tau \mid \theta_\tau = w\}$ .*
- (b) *If  $e_\phi$  is a  $T$ -fixed point in the smooth locus of  $X(w)$ , then  $e_\phi = xe_w$  for some  $x \in Q_w$ .*
- (c) *The set  $E_w^{L_w}$  has a unique minimal element given by  $e_\phi = e_{w_0(Q_w)w}$ , where  $w_0(Q_w)$  is the unique maximal element in the Weyl group of  $Q_w$ .*
- (d) *In the subdiagram of  $\hat{\mathcal{H}}_w$  (cf. Proposition 3.1.11) containing  $w := (i_1, \dots, i_d)$ , there is an edge labeled by  $s_{\alpha_m}$  for all  $m \in R_{Q_w}$  such that  $m < i_d$ .*

*Proof.* (a) follows immediately from Proposition 6.2.5 and (b) follows from (a) and Lemma 3.1.10. (c): We first note that by Lemma 3.1.16 the set of elements with head equal to  $w$  has a unique minimal element  $\tau$ . Our goal is to show that  $\tau = w_0(Q_w)w$ . If  $R'_{Q_w} = \{a_1, \dots, a_{d'_{L_w}-1}\}$  then recall the augmented sequence  $\hat{a} := (\hat{a}_1, \dots, \hat{a}_{d'_{L_w}+1}) := (0, a_1, \dots, a_{d'_{L_w}-1}, N)$ . Then  $w_0(Q_w) = x_{d'_{L_w}} \cdots x_1$  where

$$x_k = \left(s_{\alpha_{\hat{a}_k+1}} \cdots s_{\alpha_{\hat{a}_{k+1}-1}}\right) \cdots \left(s_{\alpha_{\hat{a}_k+1}} s_{\alpha_{\hat{a}_k+2}} s_{\alpha_{\hat{a}_k+3}}\right) \left(s_{\alpha_{\hat{a}_k+1}} s_{\alpha_{\hat{a}_k+2}}\right) \left(s_{\alpha_{\hat{a}_k+1}}\right) \text{ for } 1 \leq k \leq d'_{L_w}.$$

Note that  $x_k$  is the  $w_0$  of the subdiagram  $\alpha_{\hat{a}_k+1}, \dots, \alpha_{\hat{a}_{k+1}-1}$ . This means, when we consider  $w_0(Q_w)w$ , that the  $x_k$  portion acts only on those entries of  $w$  in  $\text{Block}_{L_w,k}$  and in fact can only change those entries to other values in  $\text{Block}_{L_w,k}$ . Thus we may consider  $x_k w$  individually for each  $1 \leq k \leq d'_{L_w}$ . Since  $w$  is a head,  $w \cap \text{Block}_{L_w,k}$  is either empty or maximal in  $\text{Block}_{L_w,k}$ . If  $w \cap \text{Block}_{L_w,k}$  is empty then  $x_k w = w$ . Otherwise, acting by  $x_k$  will result in  $x_k w \cap \text{Block}_{L_w,k}$  being minimal in  $\text{Block}_{L_w,k}$ . Thus  $w_0(Q_w)w \cap \text{Block}_{L_w,k}$  is either empty or minimal in  $\text{Block}_{L_w,k}$  and clearly  $\text{Class}_{w_0(Q_w)w} = \text{Class}_w$ . But this is precisely how  $\tau$  was defined in Lemma 3.1.16.

(d): We first consider the case where  $N = i_d$ . We have  $w \cap \text{Block}_{L_w,k}$  is not empty for  $1 \leq k < d'_{L_w}$ , since we have  $\hat{a}_{k+1} \in R'_{Q_w}$ , which implies, by Proposition 3.1.1, that there exists an index  $m$  with  $i_m = \hat{a}_{k+1} \in \text{Block}_{L_w,k}$ . For  $k = d'_{L_w}$  we have  $i_d = N \in w \cap \text{Block}_{L_w,k}$ .



As we saw in part (c),  $w_0(Q_w)w$  is nonempty and minimal in each  $\text{Block}_{L_w,k}$  where  $w \cap \text{Block}_{L_w,k}$  is non-empty. Thus we know, from the form of  $w_0(Q_w)$ , that in the edges that connect  $w$  to  $w_0(Q_w)w$  in  $\widehat{\mathcal{H}}_w$ , there are edges labeled by  $s_{\alpha_m}$  for all  $m \in \text{Block}_{L_w,k} \setminus \widehat{a}_{k+1}$ . But  $R_{Q_w} \cap \text{Block}_{L_w,k} = \text{Block}_{L_w,k} \setminus \{\widehat{a}_{k+1}\}$ .

Thus there is an edge labeled by  $s_{\alpha_m}$  for all  $m \in R_{Q_w}$  such that  $m \in \text{Block}_{L_w,k}$  for some  $1 \leq k \leq d'_{L_w}$ , that is for all  $m < i_d$ .

The case where  $i_d < N$  is exactly the same, except that  $w \cap \text{Block}_{L_w,d'_{L_w}} = \emptyset$ . However, since in this case the maximum value in  $\text{Block}_{L_w,d'_{L_w}-1}$  is precisely  $i_d$ , the conclusion is the same.  $\square$

**Proposition 6.2.7.** *Let  $\tau \in H_w$ . Then  $\text{mult}_\tau(w) = \text{mult}_{\theta_\tau}(w)$ .*

*Proof.* By Lemma 3.1.10 and Proposition 3.1.11 we have

$$\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \theta_\tau \text{ for some } m_1, \dots, m_t \in R_{Q_w}.$$

Setting  $x$  equal to a lift of  $s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}}$  to  $\text{GL}_N$  we have that  $e_\tau = x \cdot e_{\theta_\tau}$ , where  $\cdot$  is left multiplication. We have that  $x$  is in the stabilizer  $Q_w$  of  $X(w)$  for the action of left multiplication. This implies left multiplication by  $x$  induces an automorphism of  $X(w)$ , under which  $e_{\theta_\tau}$  is mapped to  $e_\tau$ . Thus  $\text{mult}_\tau(w) = \text{mult}_{\theta_\tau}(w)$ .  $\square$

**Proposition 6.2.8.** *Let  $\tau \in H_w$  and  $\xi \in \text{Head}_{L_w}$ . If  $\theta_\tau^{L_w,w} = \xi$  (cf. Definition 3.1.12), then  $\text{mult}_\tau(\xi) = 1$ .*

*Proof.* By Lemma 3.1.10 and Proposition 3.1.11,  $\theta_\tau^{L_w,w} = \xi$  implies

$$\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \xi \text{ for some } m_1, \dots, m_t \in R_{Q_w}.$$

We have by Proposition 3.1.5 that  $\xi \in \text{Head}_{L_w}$  is equivalent to  $R_{Q_w} \subseteq R_{Q_\xi}$ . Thus

$$\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \xi \text{ for some } m_1, \dots, m_t \in R_{Q_\xi}$$

and by Proposition 3.1.11 this implies that  $\theta_\tau^{L_\xi,\xi} = \xi$ . We conclude by Proposition 6.2.5 that  $\text{mult}_\tau(\xi) = 1$ .  $\square$

**Corollary 6.2.9.** *Let  $\tau \in H_w$  and  $\xi := (j_1, \dots, j_d) \in \text{Head}_{L_w}$ .*

- (a) *The set of  $T$ -fixed points in the smooth locus of  $X(\xi)$  contains the set  $E_\xi^{L_w} := \{e_\tau \mid \theta_\tau^{L_w,w} = \xi\}$  with equality if and only if  $R_{Q_w} \cap \{1, \dots, j_d\} = R_{Q_\xi} \cap \{1, \dots, j_d\}$ .*
- (b) *The set  $E_\xi^{L_w}$  has a unique minimal element given by  $e_\phi = e_{w_0(Q_w)\xi}$ , where  $w_0(Q_w)$  is the unique maximal element in Weyl group of  $Q_w$ .*

*Proof.* (a): By Corollary 6.2.6(a) the set of  $T$ -fixed points in the smooth locus of  $X(\xi)$  is equal to  $E_\xi^{L_\xi} := \{e_\tau \mid \theta_\tau^{L_\xi,\xi} = \xi\}$ . We have  $E_\xi^{L_w} \subseteq E_\xi^{L_\xi}$  as a trivial consequence of Proposition 6.2.8. We

would like to show that  $E_\xi^{L_w} = E_\xi^{L_\xi}$  if and only if  $R_{Q_w} \cap \{1, \dots, j_d\} = R_{Q_\xi} \cap \{1, \dots, j_d\}$ .

( $\Leftarrow$ ): Let  $\tau_1, \tau_2 \in H_\xi$  such that  $\tau_1 \leq \tau_2$  with  $\tau_1 = s_{\alpha_m} \tau_2$ . Then  $m \in \{1, \dots, j_d\}$ . Now, suppose that  $e_\tau \in E_\xi^{L_\xi}$ . Then  $\theta_\tau^{L_\xi,\xi} = \xi$ , and by Lemma 3.1.10  $\tau = s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} \xi$  with  $m_1, \dots, m_t \in R_{Q_\xi} \cap \{1, \dots, j_d\} = R_{Q_w} \cap \{1, \dots, j_d\}$ . But this implies  $\theta_\tau^{L_w,w} = \xi$ , which means  $e_\tau \in E_\xi^{L_w}$ .

( $\Rightarrow$ ): We have by Corollary 6.2.6(d) that there is a  $\tau_1, \tau_2 \in H_\xi$  with  $e_{\tau_1}, e_{\tau_2} \in E_\xi^{L_\xi}$  such that  $\tau_1 = s_{\alpha_m} \tau_2$  for all  $m \in R_{Q_\xi} \cap \{1, \dots, j_d\}$ . Then  $e_{\tau_1}, e_{\tau_2} \in E_w^{L_w}$  by our hypothesis. This implies, by Lemma 3.1.10 that

$$\begin{aligned} \tau_1 &= s_{\alpha_{m_1}} \cdots s_{\alpha_{m_t}} w \text{ for some } m_1, \dots, m_t \in R_{Q_w} \\ \tau_2 &= s_{\alpha_{n_1}} \cdots s_{\alpha_{n_s}} w \text{ for some } n_1, \dots, n_s \in R_{Q_w}. \end{aligned}$$

But we have, since  $\tau_1 = s_{\alpha_m} \tau_2$ , that

$$\tau_1 = s_{\alpha_m} s_{\alpha_{n_1}} \cdots s_{\alpha_{n_s}} w.$$

This implies that  $m_r = m$  for some  $1 \leq r \leq t$ . This follows from the fact that if  $\gamma := (j_1, \dots, j_d)$  and  $\beta := (l_1, \dots, l_d)$  with  $\gamma = s_{\alpha_{a_1}} \cdots s_{\alpha_{a_z}} \beta$  where each  $s_{\alpha_{a_i}}$  reduces the length by 1, then for some fixed  $n$ , the number of  $s_{\alpha_{a_i}} = s_{\alpha_n}$  equals  $|\{p \mid j_p \leq n\}| - |\{p \mid l_p \leq n\}|$ .

Thus  $m \in R_{Q_w}$ . Thus  $R_{Q_\xi} \cap \{1, \dots, j_d\} \subseteq R_{Q_w} \cap \{1, \dots, j_d\}$ . As Proposition 3.1.5 implies that  $R_{Q_w} \subseteq R_{Q_\xi}$  this completes our proof that  $R_{Q_w} \cap \{1, \dots, j_d\} = R_{Q_\xi} \cap \{1, \dots, j_d\}$ .

(b): This follows by the same argument made in Corollary 6.2.6(c).  $\square$

**Remark 6.2.10.** These multiplicity results have an enlightening interpretation in terms of the disjoint Hasse diagram  $\widehat{\mathcal{H}}_w$  from Proposition 3.1.11. Recall that for a degree 1 head  $\theta \in \text{Head}_{L_w}$  we have that  $\text{WStd}_\theta$  is the set of all elements  $\tau \in H_w$  such that  $\theta_\tau = \theta$ . In terms of the disjoint Hasse diagram  $\text{WStd}_\theta$  is the set of all elements that are connected to  $\theta$  in  $\widehat{\mathcal{H}}_w$ .

Thus Corollary 6.2.6 implies that  $\text{WStd}_w$ , the set of all elements connected to  $w$  in  $\widehat{\mathcal{H}}_w$ , is exactly the set of all elements whose associated  $T$ -fixed points are smooth. Proposition 6.2.7 implies that all the elements connected to the same  $\theta \in \text{Head}_{L_w}$  inside  $\widehat{\mathcal{H}}_w$  have associated  $T$ -fixed points having the same multiplicity in  $X(w)$ . Finally, Corollary 6.2.9 implies that all the elements connected to the same  $\theta \in \text{Head}_{L_w}$  inside  $\widehat{\mathcal{H}}_w$  have associated  $T$ -fixed points that are smooth in  $X(\theta)$ . Thus, considerable information about the singularities of  $X(w)$  can be inferred by inspecting  $\widehat{\mathcal{H}}_w$ .

**Example 6.2.11.** In Example 3.1.15 we constructed  $\widehat{\mathcal{H}}_w$  for  $w = (3, 6, 9)$  and  $Q_w = P_{R_{Q_w}}$  with  $R_{Q_w} = \{1, 2, 4, 5, 7, 8\}$ . We list below a few examples of the multiplicity results that may be obtained by referring to the diagram from Example 3.1.15. Both  $e_{(2,4,7)}$  and  $e_{(1,6,8)}$  are smooth points in  $X(w)$  since both  $(2, 4, 7)$  and  $(1, 6, 8)$  are connected to  $(3, 6, 9)$  in  $\widehat{\mathcal{H}}_w$ . We have  $\text{mult}_{(2,4,5)}((3, 6, 9)) = \text{mult}_{(1,5,6)}((3, 6, 9))$  since  $(2, 4, 5)$  and  $(1, 5, 6)$  are connected to the same head in  $\widehat{\mathcal{H}}_w$ . Further,  $e_{(1,3,6)}$  is a smooth point in  $X((2, 3, 6))$  since  $(1, 3, 6)$  is connected to the head  $(2, 3, 6)$  in  $\widehat{\mathcal{H}}_w$ .

Finally, we may compare the degree 1 heads  $\xi = (2, 3, 9)$  and  $\gamma = (3, 5, 6)$ . We have  $R_{Q_\xi} = \{1, 2, 4, 5, 6, 7, 8, 9\}$ . Thus, in light of Corollary 6.2.9, since  $R_{Q_\xi} \cap \{1, \dots, 9\} \neq R_{Q_w} \cap \{1, \dots, 9\}$  we have that the set of  $T$ -fixed points in the smooth locus of  $X(\xi)$  strictly contains the set of  $T$ -fixed points that are associated to the elements connected to  $\xi$  in  $\widehat{\mathcal{H}}_w$ .

However, since  $R_{Q_\gamma} \cap \{1, \dots, 6\} = R_{Q_w} \cap \{1, \dots, 6\}$ , we have that  $T$ -fixed points in the smooth locus of  $X(\gamma)$  are precisely the  $T$ -fixed points that are associated to the elements connected to  $\gamma$  in  $\widehat{\mathcal{H}}_w$ .

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